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EXAMPLES
ON
HEAT AND ELECTRICITY

H. H. TURNER

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A COLLECTION OF EXAMPLES

ON

HEAT AND ELECTRICITY

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HEAT AND ELECTRICITY

BY

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TRINITY COLLEGE.

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PREFACE.

THESE Examples are collected from the Examination Papers set in the various College Examinations and the Mathematical Tripos. Having had some trouble in obtaining a sufficient number of problems while working for Section D of Part III. of the Mathematical Tripos, I thought others might find it convenient to have them collected into this form.

TRINITY COLLEGE,
June, 1884.

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EXAMPLES ON HEAT AND ELECTRICITY.

HEAT.

Fourier's Theorem.

1. Shew that the equation

$$y=\frac{a}{2}+x-\frac{4a}{\pi^2}\left\{\cos\frac{\pi}{a}(x+y)+\frac{1}{3^2}\cos\frac{3\pi}{a}(x+y)+\frac{1}{5^2}\cos\frac{5\pi}{a}(x+y)+\&c.\right\}$$

represents a staircase of straight lines of length a, starting from the origin and parallel alternately to axes of y and x.

S. John's College, 1881.

2. Shew that $\frac{2}{\pi}\int_0^\infty \sin qx\left\{\frac{h}{q}+\tan\alpha\frac{\sin qb-\sin qa}{q^2}\right\}dq$ is the ordinate of a broken line running parallel to the axis of x from $x=0$ to $x=a$ and from $x=b$ to $x=\infty$ and inclined to the axis of x at an angle α between $x=a$ and $x=b$.

Tripos, 1883.

3. Shew that for all values of x between $-b$ and b

$$F(x)-F(-x)=\frac{2}{\pi}\int_0^\infty \sin xu\,du\int_{-b}^{+b}F(y)\sin uy\,dy.$$

S. John's College, 1881.

4. Assuming the truth of Fourier's series prove the formula

$$f(x)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(v)\cos ux\cos uv\,du\,dv,$$

where x is positive, and state any other necessary conditions. If $f(x)$ is an even function, prove

$$\left(\cosh\frac{d}{dx}\right)^n f(x)=\frac{2}{\pi}\int_0^\infty\int_0^\infty f(v)\cos ux\cos^n u\cos uv\,du\,dv.$$

Tripos, 1881.

5. Assuming Fourier's Integral

$$f(x, y)=\frac{1}{4\pi^2}\iiiint f(\xi, \eta) \cos \{\alpha(\xi-x)+\beta(\eta-y)\}\, d\xi\, d\eta\, d\alpha\, d\beta,$$

shew that in polar co-ordinates

$$f(r, \theta)=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} f(\rho, \phi)\, F(\gamma, \rho')\, d\phi\, \rho\, d\rho\, \gamma\, d\gamma,$$

where ρ' is the distance from (ρ, ϕ) to (r, θ) and

$$F(\omega)=\frac{1}{2\pi}\int_0^{2\pi} \cos(\omega \cos \lambda)\, d\lambda$$

(i.e. a Bessel's Function).

Tripos, 1880.

6. Let $\phi_1(x)$ be the integral of $\phi(x)$ taken from $x=0$ to $x=x$ and let it be such that $\phi_1(x)$ never exceeds a certain maximum value: and let $\int_{-\infty}^{\infty}\frac{\phi_1(x)}{x}\,dx$ have a value A, which is neither zero nor infinite. Prove the following generalization of Fourier's Theorem

$$f(x)=\frac{1}{2A}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \phi(zy-zx) f(y)\, dy,$$

and deduce Fourier's result.

(Liouville.)

CONDUCTION OF HEAT.

One Dimensional Problems.

7. A square prismatic bar has its ends maintained at constant temperatures, one T^0 above and the other t^0 below that of the surrounding air. If a section which divides the bar into segments whose lengths are as 2 : 1 remains at the temperature of the air, compare T and t.

Trinity Hall, 1878.

8. A uniform bar (length $2l$) of small section with an adiathermanous jacket is heated so that the three successive portions of length $\frac{2l}{3}$ are respectively at the temperatures v_1, v_2, v_3; shew that after a time

t the temperature at a distance x from the centre of the bar measured towards the end initially at temperature v_3 is

$$\frac{v_1+v_2+v_3}{3}+\frac{1}{\pi}\Sigma_{n=1}^{n=\infty}\frac{1}{n}\sin\frac{n\pi}{3}\exp\left(-\frac{n^2\pi^2kt}{c\rho l^2}\right)$$
$$\left\{2(v_3-v_1)\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{l}+(2v_2-v_1-v_3)\cos\frac{n\pi x}{l}\right\},$$

the ends being excluded.

What is the final temperature?

S. Catharine's College, 1878.

9. In linear propagation of heat given that $v=0$ when $x=0$ for all values of t, and $v=f(x)$ when $t=0$, prove that

$$v=\frac{1}{2a(\pi t)^{\frac{1}{2}}}\cdot\int_0^\infty f(\lambda)\left\{e^{-\frac{(x-\lambda)^2}{4a^2t}}-e^{-\frac{(x+\lambda)^2}{4a^2t}}\right\}d\lambda.$$

S. John's College, 1879.

10. A bar of finite length l is heated so that its two ends are at the temperature zero. If initially the temperature is given by $v=c\,.\,\frac{x(l-x)}{l^2}$, shew that the temperature at any point at the end of time v is given by

$$v=\frac{8ce-kt}{\pi^3}\left\{e^{-\frac{\pi^2kt}{l^2}}\sin\frac{\pi x}{l}+\frac{1}{3^3}e^{-\frac{3^2\pi^2kt}{l^2}}\sin\frac{3\pi x}{l}+\frac{1}{5^3}e^{-\frac{5^2\pi^2kt}{l}}\sin\frac{5\pi x}{l}+\&c.\right\}$$

the temperature of the surrounding medium being zero.

Clare College, 1879.

11. One end of an infinite rod is kept for a long time at temperature v_0, there being surface radiation. A part whose extremities are distant from this end b, $b+l$, is then cut from the rod and kept from loss or gain of heat. Shew that the temperature at time t of a point distant x from the end of the part is

$$v_0e^{-ab}\frac{1-e^{-al}}{a}+v_0ae^{-ab}\sum_{n=1}^{n=\infty}\frac{1-\cos n\pi e^{-al}}{a^2+\frac{n^2\pi^2}{l^2}}e^{-\frac{\kappa n^2\pi^2}{l^2}t}\cos\frac{n\pi x}{l},$$

where a, k are quantities depending on the nature of the rod.

Clare College, 1880.

12. A uniform bar is given with the two halves of its length at two different given temperatures V, V'. Find the temperature of any point of it at any subsequent time, the surface being so protected that there is no gain or loss of heat from without. Example, an iron bar 50 cm. long. The thermal conductivity of iron (C.G.S. units) is ·16 (water being standard substance), and its thermal capacity ·875. Prove that at 1400 secs. from the beginning the temperature at either end will be

$$\tfrac{1}{2}(V+V')+\tfrac{1}{2}(V-V')\frac{4}{\pi}\left(\frac{1}{\epsilon}-\frac{1}{3\epsilon^9}+\frac{1}{5\epsilon^{25}}-\&c.\right).$$

Tripos, 1874.

13. Find the temperature in an infinitely long bar heated at one end when the conductivity varies inversely as the absolute temperature: density, &c. being unaltered with temperature, flow of heat steady, surrounding medium at temperature zero.

Tripos, 1875.

14. In a ring heated in any manner and left to cool in a medium of temperature zero, shew that if M is the mean initial temperature, the mean temperature after time t is $Me^{-\lambda t}$, where $\lambda=\frac{hl}{CSD}$, where l and S are the perimeter and area of cross section, h exterior conductivity, C the specific heat, D the density.

Trinity College, 1874.

15. A circular ring of small section is initially at a temperature $v=F(x)$; and is surrounded by air at temperature zero; x being the arc from a fixed point. Shew that the complete solution is

$$v=\Sigma_i\left(A_i\sin\frac{ix}{r}+B_i\cos\frac{ix}{r}\right)e^{-\left(h+\frac{ki^2}{r^2}t\right)},$$

where i is a positive integer: and find A_i and B_i.

Trinity College, 1879.

16. If the temperature between the planes $x=2nc$ and $x=(2n+1)c$ be initially $(-1)^n T$, prove that after a time t the temperature at any point is

$$\frac{4T}{\pi}\Sigma_0^\infty\frac{\sin(2n+1)\pi\frac{x}{c}}{2n+1}e^{-a^2(2n+1)^2\frac{\pi^2}{c^2}t}.$$

Jesus College, 1879.

17. A homogeneous solid extending infinitely on one side of a plane, and initially at the temperature 0, radiates into a medium at temperature $f(t)$; find the temperature at any point at any time, and supposing $f(t)$ constant, express the temperature at the bounding plane by means of a tabulated integral.

Smith's Prize, 1878.

18. If the annual fluctuation of temperature at a place be expressed by $A + B\cos 2\pi\frac{t}{T}$ in degrees Centigrade, prove that the frost will reach a depth

$$\sqrt{\frac{kT}{\pi c}}\log_e \frac{B}{A}\frac{h}{\sqrt{h^2+2h\sqrt{\frac{\pi kc}{T}}+2\frac{\pi kc}{T}}},$$

where k, h are the interior and exterior conductivities and c the thermal capacity of unit volume.

Tripos, 1881.

19. If the temperature of an infinite solid have different uniform values V, V' on opposite sides of a given plane, prove that at any subsequent time t the temperature is given by the expression

$$\frac{V+V'}{2}+\frac{V-V'}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-z^2}\,dz,$$

x being measured from the plane towards the side where the temperature was initially V.

If the earth has been cooling for 200,000,000 years from a uniform temperature, and the numerical value of k be 400 (feet and years being units), prove that the rate of cooling at a depth of about 76 miles is greatest, and at a depth of about 130 miles the rate of cooling has reached its maximum value at that place for all time.

Tripos, 1881.

20. A solid is bounded by two infinite parallel planes: taking into account the radiation from its surfaces, shew that the temperature at any internal point will be given by

$$\Sigma A e^{-k^2m^2t}\cos\left[\frac{\lambda(x+a)+\lambda'(x-a)}{2a}\right],$$

where $$\lambda=\tan^{-1}\frac{l}{m}, \qquad \lambda'=\tan^{-1}\frac{l'}{m},$$

l, l' are the external conductivities of the two faces (supposed unequal) and $2a$ the thickness of the solid. The origin is taken midway between the faces and m is determined by the equation $ma=\frac{n\pi+\lambda+\lambda'}{2}$, n being any integer.

S. John's College, 1881.

21. An infinite homogeneous slab whose bounding planes are $v=\pm a$ is placed between two media, one beyond the plane $x=-a$ at which the temperature is $v_0 e^{-n^2kt}$, and the other beyond the plane $x=a$ at which the temperature is zero: shew that, if the ratio of the surface conductivity to the internal conductivity is $n\tan\beta$ and $K=kCD$ the temperature within the slab at time t is given by

$$v=v_0 e^{-n^2kt}\sin\beta\operatorname{cosec}2(\beta-na)\cos\{n(x-a)+\beta\}.$$

What would be the temperature at any point of the slab if the temperatures of the media at the two sides of the slab were $v_0 e^{-n_0{}^2kt}$ and $v_1 e^{-n_1{}^2kt}$?

Tripos, 1882.

22. Assuming that the motion of heat in a body bounded by two parallel planes perpendicular to the axis of x, and unlimited in other directions, the temperature u being a function of x only, is given by an equation of the form $\frac{du}{dt}=a^2\frac{d^2u}{dx^2}$, prove that if

$$\begin{aligned} u&=f(x) \text{ when } t=0,\\ u&=0 \quad\text{,,}\quad x=0,\\ u&=\gamma \quad\text{,,}\quad x=c, \end{aligned}$$

c being the distance between the bounding planes $x=0$ and $x=c$, and γ a constant, then

$$u=\gamma\left\{\frac{x}{c}+\frac{2}{\pi}\Sigma_1^\infty\frac{\overline{-1|}^n}{n}e^{-a^2\left(\frac{n\pi}{c}\right)^2t}\sin\frac{n\pi x}{c}\right\}$$
$$+\frac{2}{c}\Sigma_1^\infty e^{-a\left(\frac{n\pi}{c}\right)^2t}\sin\frac{n\pi x}{c}\int_0^c f(\lambda)\sin\frac{n\pi\lambda}{c}\,d\lambda.$$

Trinity College, 1878.

23. In the equation $\frac{du}{dt}=a^2\frac{d^2u}{dx^2}$ for conduction of heat shew that if $u=0$ when $t=0$ throughout the solid, and $u=\phi(t)$ when $x=0$ the solution may be put into the form

$$u=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{\infty}\phi\left(t-\frac{x^2}{4a^2\gamma^2}\right)e^{-\gamma^2}\,d\gamma.$$

In the case where $\phi(t)=e^{-t}$ ascertain the form of the solution when t is very large in the neighbourhood of the boundary plane.

Smith's Prize, 1882.

24. Assuming the interior conductivity of the earth uniform, the surface an infinite plane, the temperature at depth x at time 0 equal to μx, the surface temperature at time t equal to $\sin kt$, and the temperature at depth c constant and equal to μc, shew that the following value for temperature at time t satisfies all the conditions including the equation

$$\frac{du}{dt}=a^2\frac{d^2u}{dx^2}.$$

$$u=\frac{2}{c}\Sigma_1^\infty\exp\left(-a^2\frac{n^2\pi^2}{c^2}t\right)\sin\frac{n\pi x}{c}\int_0^c\mu\lambda\sin\frac{n\pi\lambda}{c}\,d\lambda$$
$$+\sin kt\left(\frac{c-x}{c}-\frac{2}{\pi}\Sigma_1^\infty\frac{1}{n}\sin\frac{n\pi x}{c}\right)$$
$$+\frac{2a^2\pi}{c^2}\Sigma_1^\infty n\sin\frac{n\pi x}{c}\int_0^t\sin k\lambda\,d\lambda\exp\left\{-\left(\frac{n\pi a}{c}\right)^2(t-\lambda)\right\}$$
$$+\mu c\left(\frac{x}{c}+\frac{2}{\pi}\Sigma_1^\infty\frac{\overline{-1}|^n}{n}\sin\frac{n\pi x}{c}\right)$$
$$+\frac{2a^2\pi}{c^2}\Sigma_1^\infty(-)^{n-1}n\sin\frac{n\pi x}{c}\int_0^t\mu c\,d\lambda\exp\left\{-\left(\frac{n\pi a}{c}\right)^2(t-\lambda)\right\}.$$

Trinity College, 1881.

25. A hollow sphere whose radii are a and b has its inner and outer surfaces kept at the constant temperatures θ_1 and θ_2 respectively; find the steady distribution of temperature, assuming that the conductivity varies inversely as the temperature. Draw the curves representing the distribution of temperature along a radius for the two cases $\theta_1>\theta_2$ and $\theta_2>\theta_1$.

Corpus Christi College, 1877.

26. A solid is bounded internally and externally by concentric spheres of volumes v_1 and v_2 respectively, the said surfaces being maintained at temperatures t_1 and t_2. The surfaces of equal internal conductivity are concentric spheres, the conductivity being proportional to the square of the radius. If t be the steady temperature of any surface of volume v,

$$vt(v_1 - v_2) + v_1 t_1 (v_2 - v) + v_2 t_2 (v - v_1) = 0.$$

Peterhouse, 1881.

27. If the thermal conductivity of iron be represented by $Ke^{-\frac{T}{C}}$ where the temperature is T^0 C. and K and C are constants, shew that the amount of heat flowing per second through a hollow iron shell whose internal radius is a, and external radius b, when the inner surface is maintained at T^0 C. and the outer surface at 0^0 C. is

$$4\pi K . \frac{ab}{b-a} . C\left(1 - e^{-\frac{T}{C}}\right).$$

S. John's College, 1879.

28. A uniform sphere is heated and then left to cool in a medium the temperature of which is zero: h, K are the exterior and interior conductivities, D density, C specific heat of sphere. Prove that the initial law of temperature is given by $v = C\frac{\sin nx}{x}$, where n is the least root of

$$\frac{nR}{\tan nR} = 1 - \frac{h}{k}R, \text{ and } k = \frac{K}{CD} \text{ and } R \text{ is radius of sphere.}$$

Trinity College.

29. Investigate the equation of the movement of heat in a solid sphere and obtain a solution of the case in which the sphere cools after prolonged immersion in a fluid of uniform temperature.

Corpus College, 1878.

30. A homogeneous spherical shell is heated to a uniform temperature, and then plunged into a liquid by which the temperature of the surface is kept to zero: determine the subsequent temperature at any point of the shell.

Smith's Prize, 1880.

31. A homogeneous sphere of radius a is heated so that the initial temperature is

$$v_0 \equiv Ar^{-2}\cos\theta\,(\sin mr - mr\cos mr):$$

shew that the temperature at time t is $v_0 e^{-m^2kt}$ where m is a root of the equation

$$(ah - 2K)(ma\cot ma - 1) = m^2a^2K,$$

K being the conductivity and h the surface-conductivity and the external medium at zero temperature.

Tripos, 1882.

32. The initial temperature at any point of a sphere exposed to an infinite medium at temperature zero is given by

$$\frac{1}{r}\,.\,Vc\sin(\pi r/2c)\ \exp(1 - r/c),$$

c being the radius and r the distance of the point from the centre, shew that if h be the ratio of the conductivities and λ a root of the equation

$$c\lambda\cos c\lambda + (hc - 1)\sin c\lambda = 0$$

the temperature at the same point after a time t is

$$\frac{1}{r}\Sigma_1^\infty B_m e^{-a^2\lambda_m{}^2t}\sin\lambda_m r,$$

where B_m

$$\frac{V\pi c\sqrt{\{\lambda_m{}^2c^2 + (hc-1)^2\}}\left[(hc-1)\left(1+\frac{\pi^2}{4}\right) - \lambda_m{}^2c^2(hc-3) + 2e\lambda_m c\sqrt{\{\lambda_m{}^2c^2 + (hc-1)^2\}}\right]}{\{\lambda_m{}^2c^2 + hc\,(hc-1)\}\left\{1+\left(\frac{\pi}{2}-\lambda_m c\right)^2\right\}\left\{1+\left(\frac{\pi}{2}+\lambda_m c\right)^2\right\}},$$

and $a^2 = K/CD$, K being the conductivity, C the specific heat, and D the density of the sphere.

Trinity College, 1883.

33. A solid globe of metal radius a, conductivity k_1, thermal capacity per unit volume c_1 is surrounded by one of another metal,

outer radius b, conductivity and capacity k_2 and c_2: prove that if at any time the temperatures of the two metals are represented by

$$u=\frac{\sin mr}{mr},$$

$$v=\frac{\sin ma\cos n(r-a)}{mr}+\frac{k_1 m}{k_2 n}\frac{\cos ma\sin n(r-a)}{mr}$$
$$-\left(\frac{k_1}{k_2}-1\right)\frac{\sin ma\sin n(r-a)}{mnar},$$

where $m^2\frac{k_1}{c_1}=n^2\frac{k_2}{c_2}=\theta$ suppose, the temperatures at any subsequent period will be $e^{-\theta t}u$, $e^{-\theta t}v$: and determine the equation connecting n with the surface-conductivity of the outer metal.

Tripos, 1880.

34. A solid sphere is surrounded by a concentric non-conducting spherical surface and the space between is filled with hot liquid, kept at a uniform temperature by agitation. Prove that if θ be the temperature at a point in the sphere, and θ' the temperature of the liquid at any time, then θ, θ' are of the forms

$$\theta=C+\Sigma A_n\frac{\sin nr}{r}e^{-n^2kt},\quad \theta'=C+\Sigma A_n\frac{m^2}{m^2-n^2}\frac{\sin na}{a}e^{-n^2kt},$$

where $am^2/3h$ is the ratio of heat capacity of the sphere to that of the liquid and n is any root of the equation

$$xa=\left(1+ha\frac{x^2}{m^2-x^2}\right)\tan xa.$$

Tripos, 1883.

35. Discuss the expansion of functions in series of the form

$$a_1\sin\lambda_1 x+a_2\sin\lambda_2 x+a_3\sin\lambda_3 x+\ldots$$

where $\lambda_1\lambda_2\ldots$&c. are the roots of

$$\lambda a=(1-ha)\tan\lambda a.$$

Assuming that in the previous question θ, θ' when $t=0$ can be expanded in a series of the form there indicated, investigate a method for determining the coefficients in the expansion of $\theta-\theta'$ which is supposed a given function of r.

Shew that initially if $\theta=0$ everywhere, and θ' have as in the question a constant value different from zero, then A_n will be given by

$$\left\{a+\frac{3h}{m^2-n^2}+\frac{(2+ha)\,hn^2}{(m^2-n^2)^2}\right\}A_n\sin na=\frac{2ha}{m^2-n^2}\,\theta'.$$

Tripos, 1883.

Two Dimensional Problems.

36. If two opposite sides of a rectangle are maintained at the uniform temperature T_1 the other two at the temperature T_2, express either by series or elliptic functions the permanent temperature at any internal point.

Clare College, 1881.

37. Shew that the solution of $\dfrac{d^2u}{dx^2}+\dfrac{d^2u}{dy^2}=0$, subject to the conditions $u=0$ when $x=0$ and when $x=\pi$ for all positive values of y, and $u=1$ when $y=0$ for all values of x between 0 and π, is

$$u=\frac{2}{\pi}\tan^{-1}\left(\frac{\sin x}{\sinh y}\right).$$

Transform this by the aid of conjugate functions so as to express the permanent state of temperature in a quadrant of a circle of radius a, the bounding arc being kept at temperature t_1 and the two bounding radii at equal temperatures t_2.

S. John's College, 1879.

38. An infinite plane area is divided up into equal squares like an indefinitely extended chess board: one set of squares is initially at temperature T and the others at temperature $-T$. Shew that the temperature at (x, y) at any subsequent time t is

$$T\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\pm\frac{1}{mn}\exp\left[-\frac{kt}{CD}\,\frac{(m^2+n^2)\pi^2}{4a^2}\right]\cos\frac{m\pi x}{2a}\cos\frac{n\pi x}{2a},$$

where m and n are odd integers and the upper or lower sign is to be taken as $(m+n)$ is not or is a multiple of 4, the origin being taken at the centre of one of the positive squares, and the axes parallel to its sides.

S. John's College, 1879.

39. In a uniform plate of infinite extent, E units of heat are applied at all the points $(2ma, 2nb)$ and E units abstracted at all the points $\{2ma, (2n+1)b\}$, m and n being any integers: then the permanent temperature at any point (xy) is

$$\frac{E}{4\pi R}\log\frac{1-\operatorname{cn}^2\left(K\frac{x}{a}, k\right)\operatorname{cn}^2\left(K'\frac{y}{b}, k'\right)}{1-\operatorname{dn}^2\left(K\frac{x}{a}, k\right)\operatorname{sn}^2\left(K'\frac{y}{b}, k'\right)},$$

where $\frac{K}{K'}=\frac{a}{b}$ and R is the conductivity.

Tripos, 1881.

40. Any point the radius through which makes an angle θ with a fixed radius of the edge of a circular disc of radius a is maintained at temperature $f(\theta)$ where $f(\theta+\pi)=-f(\theta)$. Shew that when the motion of heat is steady the temperature at (r, θ) is

$$\frac{2}{\pi}\int_0^\pi f(\theta+\phi)\cdot\frac{ar(a^2-r^2)\cos\phi}{a^4-2a^2r^2\cos 2\phi+r^4}\cdot d\phi,$$

the disc being supposed not to radiate heat.

Jesus College, 1881.

41. If the diameter of a circle be kept at the temperature t_1 and the circumference at temperature t_2, prove that the temperature at any point is

$$t_1+\frac{2}{\pi}(t_2-t_1)\tan^{-1}\frac{2ay}{x^2+y^2-a^2},$$

and that the isothermal lines are circles passing through the ends of the diameter which is taken as axis of x.

S. John's College, 1877.

42. A plate extends to infinity in two directions and is bounded by two straight edges which meet at right angles in A. Both edges

are at temperature 0 except a portion AB of one edge which is kept at temperature 1; prove that the temperature at any point P is

$$\frac{2}{\pi}\tan^{-1}\frac{a_2b_2}{a_1b_1} \text{ or } \frac{1}{\pi}(\angle APB - \angle APC),$$

where a_1b_1 are the axes of the ellipse, and a_2b_2 of the hyperbola which can be drawn through P having A as centre and B as focus, and where C lies in BA produced so that $AC = AB$.

Tripos, 1878.

43. If in a sector of radius a and angle α the radii be maintained at the temperature t_1 and the circumference at temperature t_2, verify that the temperature at any point of the section will be

$$\frac{2t_1}{\pi}\tan^{-1}\frac{r^{\frac{2\pi}{\alpha}} - a^{\frac{2\pi}{\alpha}}}{2a^{\frac{\pi}{\alpha}}r^{\frac{\pi}{\alpha}}\sin\pi\frac{\theta}{a}} + \frac{2t_2}{\pi}\tan^{-1}\frac{2a^{\frac{\pi}{\alpha}}r^{\frac{\pi}{\alpha}}\sin\pi\frac{\theta}{a}}{r^{\frac{2\pi}{\alpha}} - a^{\frac{2\pi}{\alpha}}}.$$

S. John's College, 1878.

44. A uniform plate is bounded by two straight edges inclined at an angle of 60^0, and is otherwise infinite. The edges are kept constantly at temperatures $\phi(\alpha)$, $\psi(\beta)$, α, β being the distances from the corner. Taking the sides of the plate as axes of co-ordinates, shew that the temperature at any point (xy) will be

$$\frac{\sqrt{3}}{2\pi}\int_0^\infty \{X\phi(\omega) + Y\psi(\omega)\}\,d\omega,$$

where $X = \dfrac{y}{y^2 + (x-\omega)^2 + y(x-\omega)} + \dfrac{x}{(y+\omega)^2 + x^2 + x(y+\omega)}$

$$- \frac{x+y}{(y-\omega)^2 + (x+\omega)^2 + (y-\omega)(x+\omega)},$$

and Y is the same function of y and x that X is of x and y.

Tripos, 1879.

45. A solid circular cylinder of infinite length is immersed in a liquid of uniform temperature, and becomes heated in such a manner

that all points at the same distance from the axis acquire the same temperature; it is then exposed to air at temperature zero: obtain the differential equation for the flow of heat, and prove that if v is the temperature at time t and distance x from the axis

$$v=e^{-mt}\mathcal{J}_0(\theta),$$

where $\mathcal{J}_0(\theta)$ is Bessel's function of degree 0, θ and m being connected by the equation

$$\theta=\frac{m}{K}\cdot CD\frac{x^2}{4}.$$

If X is the bounding radius prove that the values of m and θ are determined by the condition that θ is a root of

$$\frac{hX}{2}+K\theta\frac{d}{d\theta}\log\{\mathcal{J}_0(\theta)\}=0,$$

h being the exterior conductivity.

Trinity College, 1880.

46. A right circular cylinder of radius b and height h is initially heated in any way and then exposed in air of temperature zero, the base being permanently maintained at zero. Shew that at time t the temperature is

$$\Sigma\mathcal{J}_m(r\sqrt{\sigma^2-n^2})\left[A_{mn}\sin m\theta\sin nz+B_{mn}\cos m\theta\sin nz\right]\exp\left(-\frac{\sigma^2}{a^2}t\right),$$

where m is any integer and n, σ are determined from the equations

$$n\cos nh+c\sin nh=0,$$

$$\sqrt{\sigma^2-n^2}\,\mathcal{J}'_m(b\sqrt{\sigma^2-n^2})+\lambda\mathcal{J}_m(b\sqrt{\sigma^2-n^2})=0,$$

λ being the ratio of the external and internal conductivities and c the external conductivity. Find expressions for A_{mn} and B_{mn}.

S. Peter's College, 1880.

47. Determine the steady flow of heat in a cylindrical solid whose cross section is composed of two excentric circles.

Trinity College, 1876.

48. A plane area is bounded by a semi-ellipse and its axis major. The elliptic boundary is maintained at the uniform temperature unity

and the straight boundary at the temperature zero. Prove that the temperature at any point within the area is

$$\frac{4}{\pi}\left(\frac{\sinh\phi}{\sinh a}\sin\theta+\tfrac{1}{3}\frac{\sinh 3\phi}{\sinh 3a}\sin 3\theta+\&c.\right),$$

where $c\cosh\phi$, $c\sinh\phi$; $c\cos\theta$, $c\sin\theta$ are the semi-axes of the ellipse and hyperbola through the given point confocal with the boundary, and a is the value of ϕ at the boundary.

Tripos, 1878.

Miscellaneous.

49. The pairs of opposite faces of a parallelepiped are kept at constant temperatures a, A ; b, B ; c, C respectively ; and heat is flowing steadily through it. Shew that the lines of flow are straight : find their direction and the quantity of heat that flows through the solid in unit time.

S. John's College, 1880.

50. Prove that when the motion of heat is steady the equation $F(x, y, z, \lambda)=0$ will represent a family of isothermal surfaces, if the value of λ deduced from this equation satisfy the condition

$$\frac{\overline{\frac{d\lambda}{dx}}\Big|^2+\overline{\frac{d\lambda}{dy}}\Big|^2+\overline{\frac{d\lambda}{dz}}\Big|^2}{\frac{d^2\lambda}{dx^2}+\frac{d^2\lambda}{dy^2}+\frac{d^2\lambda}{dz^2}}=\text{a function of }\lambda.$$

Clare College, 1879.

51. Glass absorbs heat rays passing through it at a rate proportional at any point to the intensity of the radiation at the point. A plate of glass is held so that the sun's rays fall perpendicularly on it, the intensity of the radiation being I. Find the intensity J of the sun's radiation on the other side, and the temperature of the glass at any point for steady distribution, on the supposition that there is no reflection at the surfaces and that the surrounding air is everywhere at zero temperature. Shew that if x, z, y be the temperatures at the surfaces and at the middle point respectively

$$x-2y+z=a(\sqrt{I}-\sqrt{J})^2,$$

where a is a constant independent of the thickness.

Tripos, 1883.

52. If the quantity of heat per unit time given up by a particle of temperature u and mass m to one of temperature u' and mass m' at a distance r be $mm'(u'-u)\phi(r)$; prove that in an isotropic substance the heat given up in a time δt to a particle m at (xyz) by a small spherical shell of radius r and thickness δr whose centre is at (xyz) is

$$\frac{2\pi\rho m}{3} r^4\phi(r)\left\{\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right\}\delta t\,\delta r.$$

Hence prove that the conductivity of the substance is

$$\frac{2\pi\rho^2}{3}\int_0^\infty r^4\phi(r)\,dr,$$

where ρ is its density.

Clare College, 1881.

53. A body of given permanent temperature is protected from the external air of given lower permanent temperature, by a given number of spherical shells of different materials. If the sum of the volumes of the shells be given, prove that in order to keep the air in the central space as hot as possible, the shells must all be indefinitely close together and all indefinitely thin except the outer, which must be of the material for which $\frac{l}{k}-\frac{2}{h}$ is greatest, where l is the harmonic mean between its radii and h and k the conductivities.

Tripos, 1877.

54. If k=ratio of conductivities and a quantity of heat Q is instantaneously generated over a spherical surface, radius a, and left to diffuse outwards and inwards, shew that the value of v at distance r from the centre of this surface is

$$Q\frac{e^{-(r-a)^2/4kt}-e^{-(r+a)^2/4kt}}{8ar\sqrt{\pi kt}}.$$

Trinity College, 1881.

55. State Newton's law of cooling, and point out the conditions (1) respecting admissible range of temperature, (2) respecting dimensions of the solid, on which its applicability depends. For example of no. (2) consider a globe of copper and a globe of stone, each of 20 cm. diam.: conductivities of copper and stone (C.G.S. units) being ·91 and

·002 respectively: and the loss of heat from the surface of each globe being $\frac{1}{4000}$ per degree of excess temperature.

The thermal capacity of copper being ·84, find the rate per cent. of excess of temperature lost by the copper globe, and state how long it requires to go down to half the primitive excess of temperature.

Tripos, 1874.

56. In an indefinitely extended œolotropic solid the temperature at an infinite distance is zero, and over a spherical surface of radius r it is maintained constant: prove that when a permanent state is attained the surfaces of equal temperature are

$$\frac{x^2}{1+a^2\lambda}+\frac{y^2}{1+b^2\lambda}+\frac{z^2}{1+c^2\lambda}=r^2.$$

Find the temperature at any point in terms of λ.

Tripos, 1879.

57. A spherical ball of uniform material is exposed in a medium at temperature zero to the radiation of heat from an infinitely distant point.

Determine the law of permanent distribution of temperature on the surface.

Trinity College, 1878.

58. An ellipsoidal shell bounded by confocal ellipsoids has its two surfaces kept at constant but different temperatures, viz. θ_0 and θ_1. Find the stationary temperature at any point P and express the result in terms of the elliptic co-ordinates (a, a', a'') of P.

Lamé.

Thermodynamics.

59. Steam whose volume is 14 cubic feet per pound and pressure 28·8 pounds per square inch and absolute temperature 120° C. is rushing through an orifice of one square inch in section with the velocity of 1000 feet per second. Compare the kinetic energy of a pound with its thermal energy measured from that of a pound of water at 0° C.

Pembroke College, 1877.

60. Two vessels A and B are connected by a tube the volume of which may be neglected, and contain respectively weights $w_1 w_2$ of the same gas at temperatures t_1, t_2 and pressures p_1, p_2 $(p_1 > p_2)$. Communication is opened between them, gas passes from A to B until the pressures are equalised, each being p. Assuming that the vessels and tube are impervious to heat, prove that

$$\frac{w_1 t_1 + w_2 t_2}{p} = \frac{w_1 t_1}{p_1} + \frac{w_2 t_2}{p_2}.$$

Trinity College, 1878.

61. If for any substance

$$p = a + 2\beta\sqrt{t} + \gamma v$$

be the equation between pressure, volume and temperature, where a, β, γ are constants, then the difference of the specific heats at constant pressure and constant volume is constant.

S. John's College, 1879.

62. If a gas change its volume without changing its temperature the quantities of heat emitted are in A.P. while the volumes are in G.P.

Trinity College, 1877.

63. Volumes v_1, v_2 of two different gases at the same temperature are allowed to mix without doing work, so that the volume of the mixture is $v_1 + v_2$. Determine the amount of energy dissipated in the process.

Trinity College, 1878.

64. Isothermals for a perfect gas are hyperbolas. Shew that if the axis of an isothermal varies as the n^{th} power of its temperature, the difference of the specific heats will vary as the $(2n-1)^{\text{th}}$ power of the temperature. Hence shew that in nature this difference is constant.

Tripos, 1881.

65. In hydrogen the two specific thermal capacities K_p and K_v are approximately constant. Assuming their constancy, the second law of thermodynamics, and that the equation of the isothermal is

$$vp = (K_p - K_v)\,\theta,$$

prove that the entropy $\phi = K_v \log p + K_p \log v$. The capacities are supposed to be measured in absolute units of energy.

Tripos, 1882.

66. A gas is formed by mixing a number of simple gases in the proportions by mass of m_1, m_2, m_3, &c. The specific heats of these gases at constant volume and pressure are respectively c_1, k_1; c_2, k_2; &c., and the velocities of sound in them are $u_1, u_2, u_3 \ldots$: shew that the velocity of sound in the mixture is given by

$$U^2 = \frac{\Sigma(mk)}{\Sigma(mc)} \cdot \frac{\Sigma\left(\frac{mcu^2}{k}\right)}{\Sigma(m)}.$$

If the simple gases be such that $\frac{k}{c}$ is the same for all, what will this result become?

Tripos, 1878.

67. The greatest inclination of the adiabatic lines to isothermals occurs when the slope of the isothermal to the line of zero pressure is $\pi - \cot^{-1}\sqrt{\gamma}$, γ being the ratio of the specific heats at constant pressure and constant volume, each of which is assumed constant: and in the case of a perfect gas the locus of all these points of maximum angle is a straight line through the origin inclined to the line of zero pressure at an angle $\cot^{-1}\sqrt{\gamma}$.

Tripos, 1880.

68. P, P' are two points on the same isothermal on a diagram of energy for a gas: PM, $P'M'$ are perpendiculars on the axis of volumes, PDE, $P'D'E'$ two isentropics: if $P'M'$ cut PDE in D, prove that the area $PMM'D$ is equal to the area between $P'D$ and the isentropics $P'D'E'$, DE continued indefinitely.

Trinity College, 1881.

69. A perfect gas expands subject to the condition that its elasticity is always equal to the sum of that at constant temperature and that at constant entropy; prove that its specific heat, that at constant volume and that at constant pressure form a G. P.

Trinity College, 1880.

70. A body expands at constant temperature T from volume v_1 to volume v_2 against an external pressure π; prove that the amount of heat absorbed is

$$\int_{v_1}^{v_2}\left(T\frac{dp}{dt}-p\right)dv+\pi(v_2-v_1).$$

Trinity College, 1880.

71. Taking volume v of unit mass and entropy ϕ as independent variables, and denoting pressure and temperature by p and t, shew that the state of the working substance will be stable provided

$$\frac{dt}{dv}\cdot\frac{dp}{d\phi}-\frac{dt}{d\phi}\cdot\frac{dp}{dv}$$

is positive. Hence shew that the equation of the boundary line between the regions where the equilibrium is stable and unstable is given by

$$\left(\frac{dp}{dv}\right)_t=0.$$

Tripos, 1880.

72. Prove that if the working substance in an engine describing a cycle of maximum efficiency be a perfect gas, and if p, p' be the pressures at two points where the cycle meets the same isothermal, then the ratio $p:p'$ is constant.

Trinity College, 1878.

73. Prove that if $p=Rt$ where R is any function of the density of a substance, the rate of change of the internal energy per unit of heat when the pressure remains constant will be $\frac{Jk}{K}$.

Tripos, 1883.

74. Prove that if a unit mass of perfect gas expand at constant pressure p from volume v_1 to volume v_2 its temperature changing from t_1 to t_2, then farther expand without loss or gain of heat till its temperature is again t_1, and then contract at constant temperature till its volume and pressure are the same as at first, the work done during the cycle is

$$Jc(t_2-t_1)+(v_2-v_1)p-J(\phi_2-\phi_1)t_1,$$

where c is the specific heat at constant volume and $\phi_1\phi_2$ correspond to the conditions of the body at the beginning and end of its first operation.

Trinity College, 1875.

75. Prove that in a vertical column of gas in thermal equilibrium the temperature is uniform.

Tripos, 1881.

76. Find the most rapid diminution of temperature in ascending in the atmosphere that is consistent with stable equilibrium.

Smith's Prize, 1878.

77. In a thermodynamic engine the working substance passes through a cycle represented on the diagram of energy by two adiabatics AB, CD, and two curves BC, DA such that the specific heat λ is constant through each, and the same for both: $T_1 T_2 T_3 T_4$ are the temperatures at $ABCD$ respectively: find an expression for the work done in the cycle in terms of them, and prove that if $T_1 T_3$ are given the work done will be greatest if

$$T_2 = T_4 = \sqrt{T_1 T_3},$$

and that it is then $= J\lambda (\sqrt{T_3} - \sqrt{T_1})^2.$

Trinity College, 1879.

78. A uniform circular tube containing masses $m_1 m_2$ of two gases which are separated by two adiathermanous pistons, is placed in a hot fluid. Assuming the flow of heat across the substance of the tube into the gases to be at each instant the same for every point, find the temperatures of the gases when they fill given portions of the tube.

Prove that the absolute temperatures $t_1 t_2$ of the two gases are constantly connected by the relation

$$\left(\frac{t_1}{T_1}\right)^{k_1(k_2-c_2)} = \left(\frac{t_2}{T_2}\right)^{k_2(k_1-c_1)},$$

where $T_1 T_2$ are the initial temperatures, $k_1 c_1$ and $k_2 c_2$ their specific heats under constant pressure and volume.

Tripos, 1878.

79. An irregularly heated body is enclosed in a non-conducting envelope: obtain equations to determine its entropy. A cylinder closed at one end and provided with two pistons contains equal quan-

tities of the same gas at the same temperature and pressure, separated by one of the pistons. The sides and pistons are non-conductors, and the closed end which is a perfect conductor is in contact with a constant source. If the outer piston be driven in till the pressure attains a given magnitude, find the entropy of the whole gas at the end of the process.

Trinity College, 1876.

80. If two bodies of equal and constant thermal capacities (C) be at the absolute temperatures T and T' prove that the mechanical energy developed in their reduction to the same temperature θ is

$$JC(\sqrt{T}-\sqrt{T'})^2, \text{ and that } \theta^2=TT'.$$

Jesus College, 1878.

81. On the assumption that if a quantity H of heat be instantaneously communicated to one point of an infinite homogeneous isotropic solid at absolute temperature v_0 throughout, the temperature at distance r from the point at time t will be

$$v_0+\frac{H}{8}\left(\frac{c}{\pi kt}\right)^{\frac{3}{2}}e^{-\frac{cr^2}{4kt}},$$

where c is the specific heat, k the conductibility; shew that the available energy in heat form is

$$\frac{cH^2}{16v_0}\left(\frac{c}{2\pi kt}\right)^{\frac{3}{2}}.$$

Tripos, 1883.

82. A mass of ice at 0^0C. is subjected to a pressure of 40 atmospheres, without being allowed to give or receive heat. Given that the specific heat of ice at constant pressure is about half that of water, that the latent heat of melting is 79, and freezing point is lowered ·0075 of a degree per atmosphere; shew that rather less than $\frac{1}{500}$th of the mass will be melted.

Tripos, 1877.

83. If λ be the latent heat of evaporation of a liquid at temperature t and ss' the specific heat of the liquid and the vapour respectively at the point of saturation, prove that

$$\frac{d}{dt}\frac{m\lambda}{t}+\frac{s}{t}=0,$$

where the liquid and vapour are contained in a non-conducting cylinder and are allowed to expand by means of a piston : m being the ratio of the mass of vapour to whole mass.

Trinity College, 1882.

84. A vessel partly filled with water and partly with steam has a manometer attached. It is required to graduate the manometer so as to indicate the temperature in the thermodynamic scale. Shew that if w, s be the volumes of unit masses of water and of steam at temperature θ, and pressure p: k the latent heat corresponding to the passage of unit mass of water into steam: and J Joule's equivalent: p_0, θ_0 a standard pressure and temperature

$$\log \frac{\theta}{\theta_0} = \int_{p_0}^{p} \frac{(s-w)\,dp}{Jk},$$

and explain how when the results of a series of experiments for determining w, s and k at various pressures are given, the manometer may be properly graduated.

Tripos, 1882.

85. A unit mass of steam and water of which a fraction x_1 is originally steam is brought without communication or subtraction of heat from the absolute temperature θ_1 to θ. If x is the fraction which is now in the state of steam and if ξ_1, ξ denote the latent heats of evaporation of a pound of steam at these temperatures, shew that

$$\frac{x\xi}{\theta} = \frac{x_1\xi_1}{\theta_1} + c \log_e \frac{\theta_1}{\theta},$$

c being the specific heat of water at constant pressure, supposed constant.

S. Peter's College, 1876.

86. Out of a unit mass of water and steam a fraction x is in the state of steam. Shew that the specific heat of the mixture kept always at the point of evaporation differs from its specific heat at constant volume by

$$\frac{t}{J} \cdot \frac{dp}{dt_{v\text{ const}}} \times \frac{dv}{dt_{x\text{ const}}}.$$

S. Peter's College, 1878.

87. A pound of water and a pound of ice are enclosed at the same temperature in a vessel impervious to heat. If the pressure is increased trace the motion of the point representing the state of the mixture. When will the point reach the thermodynamic model, and will its path touch or cut it?

Clare College, 1880.

88. Prove that if the temperature of a mixture of liquid and vapour differ from the critical temperature by a small quantity θ', the latent heat of the vapour will be $2c\theta'$.

Trinity College, 1881.

89. A piston without weight in the middle of a closed cylinder separates a volume v of air at pressure $3p$ and temperature t, from a volume v of steam at its saturating density at temperature t, its pressure being $2p$. The cylinder is a perfect conductor and is immersed in a quantity of liquid infinite compared with its contents and at temperature t. If the piston having been at first fixed is released, find the energy dissipated before the position of equilibrium is attained.

Trinity College, 1881.

90. The freezing-point of ice is lowered by 1° F. for an increase of pressure of 1125 pounds per square inch. A beam of ice at 30° F. one inch wide two inches deep and two feet long is supported at each end. Shew that the greatest load which can be uniformly distributed upon it is 500 lbs., assuming that the neutral line passes through the centre of the beam.

S. John's College, 1879.

91. If unit mass of a mixture of steam and water be allowed to expand under a pressure always equal to its own elasticity, prove the equation

$$W = d(pv) - \frac{Elx}{t} \, . \, dt - u\,dp,$$

where W is external work done, x the mass of steam at any instant in the mixture, and v is the volume of the mixture.

Apply this to prove that the useful work done by unit mass of steam in one stroke of an ordinary condensing engine furnished with a steam jacket between leaving and re-entering the boiler, is

$$E\,.\int_{t_1}^{t_2}\frac{L}{t}\,dt,$$

where t_1 and t_2 are temperatures of condenser and boiler.

Tripos, 1877.

92. P, P' are the vapour tensions of a substance when the difference of the pressures on the surface is p, p being large compared with P, v the volume in the denser state and at pressure P, V the volume of whole as a vapour at that pressure and k the compressibility; prove that neglecting small terms of the order Pk

$$\frac{P'}{P}=e^{\frac{pv}{PV}\left(1-\frac{pk}{2}\right)}.$$

Trinity College, 1881.

93. Shew that if v, v' are the volumes of the same mass of water and ice at temperature $-t^0$, ϖ, ϖ' the maximum vapour tensions of water and ice at that temperature, V the volume of the vapour at pressure ϖ, k, k' the coefficients of cubical compressibility of water and ice and p the pressure required to lower the melting point to $-t^0$, then neglecting terms of the order ϖk and ϖv

$$\varpi V\log\frac{\varpi}{\varpi'}=p\left\{v'\left(1-\frac{pk'}{2}\right)-v\left(1-\frac{pk}{2}\right)\right\}.$$

Trinity College, 1883.

94. Find the work done in one stroke of a compound engine in which one piston-rod passes through the high and low pressure cylinders, when the steam in the high pressure cylinder is cut off at a given fraction of the stroke, supposing the steam to expand adiabatically.

Tripos, 1881.

95. A small length a at each end of the cylinder of an engine whose total length is $2a+h$, is not traversed by the piston. The steam is cut off after the piston has travelled a length x of the cylinder, and when it gets to a distance $a+x$ from the other end the steam is let in at the other side and "cushions" against the piston for the remainder

of the stroke. If the cylinder be of non-conducting material shew that in order to obtain the maximum work per stroke $a+x$ must be equal to $(a+h)\,\gamma^{1-\gamma}$ where γ is the ratio of the specific heats at constant pressure and volume and the steam is supposed to expand as a perfect gas.

Deduce the corresponding result if the cylinder have a steam-jacket.

S. Peter's College, 1880.

96. A freezing machine (Giffard's) was exhibited at the Paris exhibition. It consists of two cylinders the pistons of which work on to two cranks on the same shaft, driven by an external source of power, and of a large air reservoir which is always maintained at the temperature of the outside air. In the first cylinder air is compressed till its pressure is the same as in the reservoir, when valves open and the air passes, as the stroke is completed, into the reservoir. The second and smaller cylinder acts as an engine receiving compressed air from the reservoir for such a portion of the stroke that being expanded for the remainder of the stroke it is discharged at atmospheric pressure but at a lower temperature. If V_1 and V_2 be the volumes of the cylinders, and if the compression and expansion be supposed adiabatic, prove that the work done during each stroke in the first cylinder is

$$\Pi V_1 \frac{\gamma}{\gamma-1} \frac{V_1-V_2}{V_2},$$

and in the second cylinder is

$$\Pi \frac{\gamma}{\gamma-1} (V_1-V_2),$$

Π being the atmospheric pressure.

Tripos, 1879.

ELECTROSTATICS.

97. What is the cyclomatic number of the space

(*a*) inside an anchor ring?

(*b*) outside an anchor ring?

(*c*) outside two linked anchor rings?

98. If ξ, η, ζ be components of a vector $\nabla\Psi$ find X, Y, Z so that

$$\xi=\frac{dZ}{dy}-\frac{dY}{dz}, \text{ \&c.}$$

(1) when $\Psi=\dfrac{x}{x^2+y^2}$ (assume X, $Y=0$),

(2) when $\Psi=\dfrac{x}{r^3}$ (assume $X=0$),

and hence find the integral $\iint(l\xi+\dots)\,dS$ by means of a line integral in the second case over the portion of the surface $r=a$ cut off by $x=b$.

99. If there be two concentric spherical shells insulated from one another, and they be connected by a wire the inner one is wholly discharged. If the law of force was $\dfrac{1}{r^{2+p}}$ prove that there would be a charge B on the inner shell such that if A were the charge on the outer shell, and f, g the sum and difference of the radii

$$2gB=-Ap\,\{(f-g)\log(f+g)-f\log f+g\log g\}$$

approximately.

Tripos, 1877.

100. The position of a point in space being given by

$$x=(c+r\cos\theta)\cos\phi,\ y=(c+r\cos\theta)\sin\phi,\ z=r\sin\theta,$$

form Poisson's equation with r, θ, ϕ as variables. In the case of a symmetrical distribution about axis of z, prove that Laplace's equation is satisfied by

$$V=C\left(\frac{1}{r}-\frac{c}{r^2}\cos\theta+\ldots+(-1)^n\frac{3\cdot5\ldots(2n-1)}{\lfloor\underline{n}}\frac{c^n\cos n\theta}{r^{n+1}}\right),$$

provided the series is convergent.

Hence by summation of the series prove :—Let there be a curve for which $rr'=a^2(1+\cos\psi)$, and let this curve generate a surface of revolution about an axis bisecting AB at right angles; then if the surface so described be that of a conductor electrified and uninfluenced the equipotential surfaces are of the same kind.

Trinity College, 1876.

101. Shew that if in a uniform mass conducting electricity the potential be zero over the plane xy, its value at neighbouring points out of that plane will be

$$z\chi-\frac{z^3}{1\,.\,2\,.\,3}\nabla^2\chi+\frac{z^5}{\lfloor\underline{5}}\nabla^4\chi-\&c.,$$

where χ is a function of x and y and

$$\nabla^2\equiv\frac{d^2}{dx^2}+\frac{d^2}{dy^2}.$$

If a neighbouring equipotential surface $z=z_1$ coincide with a plane parallel to xy except over a certain finite region where there is a slight deviation the conductivity between the equipotential surfaces $z=z_1$ and $z=0$ is expressed approximately by

$$\iint\left\{1+\tfrac{1}{3}\left(\left(\frac{dz_1}{dx}\right)^2+\left(\frac{dz_1}{dy}\right)^2\right)\right\}\frac{dx\,dy}{z_1},$$

the area of integration including the whole of the above-mentioned region.

Tripos, 1876.

102. An electrified body is placed in the vicinity of a conductor in the form of a surface of anticlastic curvature. Shew that, at that point on any line of force passing from the body to the conductor where the force is a minimum, the principal curvatures of the equipotential surface are equal and opposite.

Tripos, 1883.

103. The surface of a conductor being one of revolution whose equation is $\frac{4}{r}+\frac{1}{r'}=\frac{7}{12}$, where r, r' are the distances of any point from two fixed points at distance 8 apart, find the electric density at either vertex when the conductor has a given charge.

Trinity College, 1876.

104. Given that the lines of force in any field are a series of equal equiangular spirals, find the equipotential surfaces and the law of force.

Corpus Christi College, 1877.

105. A unit of positive electricity is placed at the origin of co-ordinates and is surrounded by a dielectric of which the specific inductive capacity at any point is $1+Cf(x, y, z)$, where C is small and $f(x, y, z)$ is positive for all values of x, y, z and does not become infinite: shew that the apparent electrical density at any point will be

$$-\frac{C}{4\pi\gamma^3}\left(x\frac{df}{dx}+y\frac{df}{dy}+z\frac{df}{dz}\right),$$

where squares and higher powers of C are neglected.

Clare College, 1879.

106. Two Leyden jars A, B have capacities C_1C_2 respectively. A is charged and a spark taken, it is then charged as before and a spark passed between the knobs of A and B; A and B are then separated, and each discharged by a spark: shew that the energies of the four sparks are as

$$(C_1+C_2)^2 : C_1^2 : C_1C_2 : (C_1+C_2)\,C_2.$$

Clare College, 1880.

107. Shew that if a given charge be distributed over a surface, the intrinsic energy is a minimum when the distribution makes the given surface an equipotential.

S. John's College, 1879.

108. Two infinitely thin equal concentric circular discs are in contact and charged with electricity: find the amount of energy expended in separating them and removing one of them to an infinite distance.

Tripos, 1876.

109. If a conductor C, which entirely surrounds a system of charged and insulated conductors, be at first insulated and at potential V, and then put to earth, the potentials of all the interior conductors will be diminished by V. If this system be now discharged, the loss of energy is the same as if C had not been put to earth, but had had the interior conductors put into connexion with its inner surface.

Tripos, 1882.

110. If the capacities of two bodies are A, a, when they are each at an infinite distance from any other body, shew that when they are at a distance R, great compared with their dimensions, their capacities are increased in the ratio $R^2 : R^2 - Aa$.

Tripos, 1878.

111. Three conductors whose capacities are a, β, γ when insulated in open space are placed at points A, B, C whose distances from each other are very large compared with the dimensions of any one, shew that A's coefficient of capacity becomes

$$\frac{ab^2c^2(a^2-\beta\gamma)}{a^2b^2c^2+2a\beta\gamma abc-\beta\gamma b^2c^2-\gamma ac^2a^2-a\beta a^2b^2},$$

where a, b, c are the sides of the triangle ABC. Investigate the corresponding expression for n conductors.

Tripos, 1881.

112. At equal distances round the circumference of a circle radius r are placed n conductors each of capacity p in open space. Shew that the capacity of each is

$$p\left[1+\frac{r^2}{p^2}\sum_1^{n-1}\frac{1}{\sin^2\frac{s\pi}{n}}\right]$$

neglecting powers of $\frac{r}{p}$ above the second.

113. There are two infinite plane conductors at potentials V_1, V_2: the medium between them consists of two parts of thicknesses a and b, whose specific inductive capacities are K_1 and K_2: find the

potential at any point between the planes, and shew that the apparent charge on the surface of separation of the dielectrics is

$$\pm\frac{(V_1-V_2)(K_1-K_2)}{4\pi(K_2a+K_1b)},$$

and explain the double sign.

Jesus College, 1878.

114. A long rod uniformly charged is placed perpendicularly to a large conducting plane and with an end nearly in contact with the plane: shew that if the plane be put in connexion with the earth, the density of the electricity induced on the plane will vary inversely as the distance from the rod.

Clare College, 1880.

Spheres, &c.

115. A spherical soap-bubble of radius a being electrified to a potential P just sufficient to keep the bubble in equilibrium when the air within and without is at the same pressure: prove that the tension which the liquid film exerts across a line of unit length $=\dfrac{P^2}{16\pi a}$.

S. John's College, 1875.

116. Shew that the capacity of a conducting sphere, insulated within a concentric hollow conducting sphere, is a fourth proportional to the distance between the surfaces of the spheres and the two radii.

Tripos, 1873.

117. A condenser consisting of a conducting sphere enclosed in a concentric spherical shell, having a given charge is gradually discharged by first connecting the outer with the earth, then insulating the outer and connecting the inner with earth and so on. Shew that after the first operation the quantities of energy subsequently lost form a geometric series. Verify the result by summation.

S. Peter's College, 1879.

118. A condenser is formed of two spherical conducting sheets and has between them a conducting shell of a dielectric substance

different from the rest of the space between. Shew that if C be its capacity,

$$\frac{1}{C}=\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)\frac{1}{k_1}+\left(\frac{1}{r_2}-\frac{1}{r_3}\right)\frac{1}{k_2},$$

where r_1, r_2, r_3, r_4 are the radii of the four surfaces in ascending order, k_1 the S.I.C. of the media next the conductors, and k_2 that of the shell.

Tripos, 1883.

119. The space between two concentric spherical surfaces whose radii are a and b and which are kept at potentials A and B, is filled with a heterogeneous dielectric whose specific inductive capacity varies as the n^{th} power of the distance from their common centre. Supposing that there is no volume distribution of electricity, shew that the potential at any point between the surfaces is

$$\frac{Aa^{n+1}-Bb^{n+1}}{a^{n+1}-b^{n+1}}-\frac{a^{n+1}b^{n+1}}{r^{n+1}}\cdot\frac{A-B}{a^{n+1}-b^{n+1}}.$$

Clare College, 1879.

120. Two spherical conductors each of radius r are respectively insulated within two other concentric conductors each of radius R and insulated; charges ee' are given to the inner surfaces, and

(1) the first inner surface is connected with the second outer surface, then

(2) the second inner surface with the first outer, then

(3) each inner with its corresponding outer.

Shew that the charges on the two outer surfaces are now

$$(e+e')\,\frac{2Rr}{(R+r)^2}+e'\cdot\frac{R-r}{R+r},\ \text{and}\ (e+e')\,\frac{R^2+r^2}{(R+r)^2}-e'\cdot\frac{R-r}{R+r},$$

respectively.

S. John's College, 1880.

121. A sphere of radius a with a charge E is influenced by a charge E' on another sphere at distance d, so large that the force due to it may be considered uniform in the neighbourhood of the first sphere. Shew that that part of the energy of the first sphere which is due to the presence of the second is

$$\frac{EE'}{d}-\frac{a^3E'^2}{d^4}.$$

S. Peter's College, 1881.

122. If a sphere of radius a is introduced into a uniform field of force, find the potential due to the sphere and the density at any point.

123. If a sphere of radius a be introduced into a field whose potential is $Ayz+Bzx+Cxy$, the potential at any point outside the conductor is

$$\frac{Q}{r}-\frac{a^5}{r^5}(Ayz+Bzx+Cxy),$$

and find the equations of the line of no electrification.

Tripos, 1882.

124. A solid globe of glass is introduced into a field of uniform force X parallel to the axis of x. Shew that the potential inside the globe is

$$-\frac{3}{K+2}Xx,$$

where K is the S.I.C. of glass.

125. A nearly spherical conductor the equation of whose surface is

$$r=a(1+\lambda_1+\lambda_2+\ldots),$$

where λ_1, λ_2 ... are surface harmonics whose squares may be neglected, is surrounded by an uninsulated spherical concentric shell radius b. If the charge of the conductor is Q, prove that the density at any point of it is

$$\frac{Q}{4\pi a^2}\left(1+\frac{3a^3}{b^3-a^3}\lambda_1+\frac{4a^5+b^5}{b^5-a^5}\lambda_2+\frac{5a^7+2b^7}{b^7-a^7}\lambda_3+\&c.\right).$$

Clare College, 1881.

126. A conductor is formed by two surfaces, the inner a sphere whose equation is $r=a$, and the outer a spheroid whose equation is $r=b(1+\epsilon S_2)$, where S_2 is a surface harmonic of the second order, the interspace being a dielectric whose S.I.C. is K. Prove that when

charged so that the difference of potential at the surfaces is unity, the electrification at any point of the inner surface is, neglecting ϵ^2,

$$\frac{Kab}{4\pi(b-a)}\left(\frac{1}{a^2}-\frac{5ab^2\epsilon S_2}{b^5-a^5}\right).$$

Jesus College, 1881.

127. A point of electricity e is placed on the prolongation of the axis of the conductor $r=a(1+\epsilon\cos\theta)$ at a distance f from the pole. The conductor, which differs slightly from a sphere, is insulated and has a given charge E. Shew that if U be the potential at any internal point on the supposition of the conductor being exactly spherical, the actual potential is

$$U+A_1\left(\frac{a}{r}\right)^2 P_1+\ldots+A_n\left(\frac{a}{r}\right)^n P_n+\ldots,$$

where $$A_n=-\frac{e\epsilon a^{n-1}}{f^n}\left\{n+\frac{a^2}{f^2}(n+1)\right\},$$

except when $n=1$; and find A_1.

S. John's College, 1881.

128. If two concentric spherical surfaces, radii a and b, be slightly distorted into the surfaces

$$r=a(1+\epsilon S_i) \text{ and } r=b(1+\eta S_i),$$

prove that the change of electrification at any point of the inner surface is

$$\frac{\sigma}{4\pi}\cdot\frac{b}{a}\cdot\frac{(2i+1)\,a^{i+1}\,b^i\eta-\{(i+2)\,a^{2i+1}+(i-1)\,b^{2i+1}\}\,\epsilon}{(b-a)(b^{2i+1}-a^{2i+1})}\,S_i,$$

where σ is the original surface density.

S. John's College, 1881.

129. Find the surface density on a spherical conductor of zero potential under the action of two equal external electrified points situate at equal distances on opposite sides of the centre.

Tripos, 1876.

130. A sphere made of a substance whose specific inductive capacity is K is placed in air, and quantities e, e' of electricity are placed at two points P and Q respectively inside and outside the sphere and

on the same radius. If a be the radius of the sphere, c and d the distances of P and Q from the centre, shew that the force on the electrified particle at Q is

$$\frac{ee'}{(d-c)^2}+\frac{1-K}{1+K}\frac{2}{d^2}\left\{\frac{ee'}{\left(\frac{c}{d}\right)^{\frac{1}{1+K}}}\int_0^{\frac{c}{d}}\frac{x^{\frac{1}{1+K}}}{(1-x)^3}dx+\frac{e'^2a}{d\left(\frac{a^2}{d^2}\right)^{\frac{1}{1+K}}}\int_0^{\frac{a^2}{d^2}}\frac{x^{\frac{1}{1+K}}}{(1-x)^3}dx\right\}.$$

Trinity College, 1883.

Ellipsoids.

131. If the uninsulated ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be placed in a field of force of which the potential is Myz, prove that the electrification at any point of its surface is

$$\frac{Myz}{4\pi abc}\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{d\phi}{ab^2}-\frac{d\phi}{dc^2}}p, \text{ where } \phi=\int_0^\infty\frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

Tripos, 1881.

132. A condenser consists of two confocal ellipsoids whose equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1.$$

Shew that its capacity is

$$\frac{2}{\int_0^k\frac{dk}{\sqrt{(a^2+k)(b^2+k)(c^2+k)}}},$$

the dielectric being air.

If the dielectric be arranged in ellipsoidal shells confocal with the conductors and such that the specific inductive capacity of each shell is inversely proportional to its content, shew that the capacity is

$$\frac{2Kabc}{k},$$

where K is the S. I. C. of the innermost layer.

S. John's College, 1879.

133. The opposing surfaces of an electric accumulator are similar and similarly situated concentric ellipsoids of semiaxes a, b, c, and $a(1+\lambda)$, $b(1+\lambda)$, $c(1+\lambda)$, where λ is small: prove that the capacity is

$$\frac{1}{3}\left(\frac{1}{\lambda}+1\right)abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

Tripos, 1882.

134. Shew that the capacity of a prolate spheroid in an infinite field is $\frac{c}{\gamma}$, when the axes are $2c \coth \gamma$ and $2c \operatorname{cosech} \gamma$.

S. John's College, 1878.

135. In a charged spheroid prove that the quantity of electricity on any part of the surface cut off by two planes perpendicular to the axis is proportional to the distance between the planes.

S. John's College, 1877.

136. Calculate the "coefficients of potential" in the case of two conductors whose surfaces are confocal ellipsoids of revolution, the squares of whose semi-axes are

$$(a^2+c^2,\ a^2+c^2,\ a^2), \text{ and } (A^2+c^2,\ A^2+c^2,\ A^2).$$

S. John's College, 1881.

137. Investigate the form of Green's Function for a conductor in the form of a prolate spheroid, under the action of an unit of electricity at one of the foci.

Smith's Prize, 1881.

138. An infinite elliptic cylinder is kept at zero potential in a field whose potential is Axy. Shew that the density of the induced charge is

$$-A\,\frac{(a+b)^2}{4\pi a^2 b^2}\,pxy.$$

S. John's College, 1881.

139. An infinite conducting elliptic cylinder is placed in a field of electric force of which the potential is $-(Ax+By)$, where the axes of x and y are the axes of any section of the cylinder, and A, B are con-

stants. Shew that the potential of the electricity induced on the cylinder at any external point is

$$\frac{e^{2\alpha_0}+1}{e^{2\alpha}+1}Ax+\frac{e^{2\alpha_0}-1}{e^{2\alpha}-1}By,$$

where $c\cosh\alpha_0$ is the semi-major axis of the section of the cylinder, and $c\cosh\alpha$ that of a confocal through the point (x, y). Find the density of electricity at any point.

Jesus College, 1879.

Cylinders.

140. Two equal conducting cylinders are placed with their axes parallel at distance r. One is charged with quantity E per unit length, and the other is put to earth and then insulated. Shew that if d be the diameter of either the potential energy will be

$$2E^2\log\frac{r+\sqrt{r^2-d^2}}{d}$$

per unit length.

Hence find the force between them and draw the lines of force.

Tripos, 1883.

141. An infinite conducting cylinder is placed with its axis parallel to an uninsulated infinite conducting plane, and at a distance c from it. The cylinder is maintained at potential V, shew that the charge per unit of length is

$$Q=\tfrac{1}{2}V\div\log\frac{c+\sqrt{c^2-a^2}}{a},$$

and that the surface density is inversely proportional to its distance from the plane.

S. Peter's College, 1880.

142. Two infinite cylinders whose axes are parallel and at a distance d form a condenser. Prove that the capacity of length l is

$$\tfrac{1}{2}l\left\{\log\tan\frac{\theta_2}{2}-\log\tan\frac{\theta_1}{2}\right\}^{-1},$$

where the radii of the cylinders are r_1 and r_2 and

$$\tan\theta_1 = \frac{r_1}{r_0},\ \tan\theta_2 = \frac{r}{r_0},$$

and r_0 satisfies

$$4r_0^2 d^2 = (r_1 + r_2 + d)(r_1 + r_2 - d)(r_2 - r_1 + d)(r_2 - r_1 - d).$$

Deduce the corresponding expression for coaxial cylinders.

S. Catharine's College, 1878.

143. A condenser is formed by two long conducting circular cylinders one of which is entirely inside the other: if r r' be the radii, d the distance between the axes, and $2a$ the distance between the limiting points of the coaxial system obtained by a cross section, prove that the inverse of the capacity per unit length is

$$2 \log \frac{r'(r'^2 - r^2 - d^2 + 2ad)}{r(r'^2 - r^2 + d^2 + 2ad)}.$$

S. John's College, 1881.

144. A metal plate is intersected at right angles in the straight line AOD by an infinitely thin and infinitely extended plane conductor, and it is also pierced at right angles by two indefinitely long cylinders which cut it in equal circles whose centres are B and C, where BC is at right angles to and bisected by AOD in the point O. If O be origin and BC axis of x, and the plane conductor be kept at the constant potential V_1 and the cylinders at any two constant potentials equal to each other, prove that the potential V at any point P of the plate will be given by the equation

$$V = V_1 + kE \log \frac{\rho}{\rho'} + (1 - k) E \log \frac{\rho'}{\rho},$$

where ρ and ρ' are the distances of P from two points on BC equidistant from O, and k is a quantity equal to O when x is negative and equal to 1 when x is positive and E is a constant.

Prove also that the lines of flow and lines of equal potential on either side of AOD will form two orthogonal systems of circles.

Tripos, 1877.

145. An infinite circular cylinder of radius a is maintained at potential zero and is surrounded by two semi-cylinders of radius b,

which together make a cylinder coaxial with the former: one half is at potential A and the other half at B. Shew that the potential at all points within them is

$$\frac{A+B}{2}\frac{\log r - \log a}{\log b - \log a}$$
$$+\frac{2}{\pi}(A-B)\Sigma_1^\infty \frac{1}{2n+1}\left\{\frac{\left(\frac{r}{a}\right)^{2n+1} - \left(\frac{a}{r}\right)^{2n+1}}{\left(\frac{b}{a}\right)^{2n+1} - \left(\frac{a}{b}\right)^{2n+1}}\right\}\sin(2n+1)\theta.$$

S. John's College, 1880.

146. A very long cylindrical conductor of small section charged with electricity whose amount per unit of length is given is placed in the acute angle between two plane conductors kept at potential zero, so that the axis of the cylinder is parallel to the intersection of the planes: if the planes intersect at an angle $\frac{\pi}{n}$, find the density of the induced electricity in either plane: also the points where the density is greatest when the cylinder is placed symmetrically between the planes.

Tripos, 1879.

147. Prove that for the transformation

$$(x'+iy')^n = a^{n-1}(x+iy)$$

the densities at corresponding points in the new and old systems are in the ratio

$$n^2\left(\frac{x^2+y^2}{a^2}\right)^{\frac{n-1}{n}} : 1.$$

By means of this transformation, from the known distribution of electricity on an infinite plane conductor at potential zero in the presence of an uniformly charged infinite rod placed parallel to it, obtain the distribution of electricity upon a cylindrical conductor whose equation is

$$r^n \cos n\theta = a^n,$$

maintained at potential zero in the presence of n infinite rods arranged symmetrically as generators round the cylinder $r=b$. Find also the potential due to the distribution.

Clare College, 1881.

148. If electricity be distributed in equilibrium over a cylindrical surface whose trace on any plane perpendicular to its axis is determined by the relation $SP \,.\, HP = c^2$, where S, H are two fixed points, and P any point on the section of the surface, prove that the surface density at P is proportional to CP, where C is the middle point of SH.

Clare College, 1878.

149. Electricity is distributed in equilibrium over the surface of a right cylinder of infinite length the form of whose transverse section is $x^4+y^4=a^4$. Prove that its attraction on any external point (r, θ) is proportional to

$$\left(r^4+2a^4\cos 4\theta+\frac{a^8}{r^4}\right)^{-\frac{1}{4}},$$

and that its direction makes with the axis of x an angle

$$\tfrac{1}{2}\tan^{-1}\left(\frac{r^4-a^4}{r^4+a^4}\tan 2\theta\right)^{\frac{1}{2}}.$$

Clare College, 1880.

Inversion.

150. An electrified point P having a charge e is between two concentric spherical surfaces maintained at potential zero: if O be the common centre, and if $OAPB$ be the radius vector cutting the inner surface in A, and the outer in B, prove that the induced charges on these surfaces are

$$-e\,.\,\frac{OA\,.\,BP}{OP\,.\,AB} \text{ and } -e\,.\,\frac{OB\,.\,AP}{OP\,.\,AB}.$$

S. John's College, 1881.

151. Two particles e and e' are placed in mediums of S.I.C. K_1 and K_2 respectively, the surface of separation being a plane which bisects at right angles the distance (r) between the particles. Shew that the force on e is

$$\frac{e\left\{2e'+e\left(1-\frac{K_2}{K_1}\right)\right\}}{(K_1+K_2)\,r^2}.$$

Jesus College, 1881.

152. An insulated spherical conductor under the action of an external electrified point has the potential which the point would produce at the distance of its centre. Find the total charge and the line of no electrification.

S. Peter's College, 1879.

153. An insulated sphere of radius a is under the influence of an electrified point e at a distance f from the centre $(f > a)$: if the whole charge on the sphere is $-e\frac{a^3}{f^3}$, find the position of the line of no electrification, and the charge on each side of this line.

Tripos, 1875.

154. A small light body is charged with electricity and is suspended by a silk fibre from a point vertically above the centre of a conducting insulated sphere whose whole charge is nil; find the time of oscillation of the small body.

Trinity College, 1878.

155. M, M' are two points external to a sphere and on the same diameter, t, t' are the lengths of the tangents from M, M' to the sphere, which is at potential zero. Charges proportional to t and $-t'$ respectively are placed at M, M'. Shew that the resultant electrical force is zero at all points in which the sphere is met by a sphere of radius $MM' \cdot \frac{tt'^2}{t^2 - t'^2}$, of which the centre O is a point on MM' such that

$$OM' = MM' \cdot \frac{t'^2}{t^2 - t'^2}.$$

Trinity College, 1881.

156. Two insulated conducting spheres are charged with electricity of the same kind: supposing for simplicity the charge on one to be feeble, find the distance between them beyond which there is repulsion, and within which there is attraction.

Can you suggest an explanation of the fall of rain which is often found to succeed a flash of lightning?

Smith's Prize, 1875.

157. An infinite plane with a projecting hemispherical boss is introduced into a field of electrical force of uniform density F whose

direction is perpendicular to the plane. Shew that the surface density of the induced electricity is (i) at any point of the boss $\frac{3}{4\pi} F \cos\theta$, (ii) at any point of the plane $\frac{1}{4\pi} F \frac{r^3 - a^3}{r^3}$, r being the distance of the point from the centre of the hemisphere and a the radius of the hemisphere.

Tripos, 1882.

158. An electrified point e is placed between two infinite plane faces of a conductor which cut orthogonally: find the distribution of the induced electricity, and prove that the whole amounts on the faces are

$$-\frac{2e}{\pi}\tan^{-1}\frac{b}{a} \text{ and } -\frac{2e}{\pi}\tan^{-1}\frac{a}{b}$$

respectively, a and b being the perpendicular distances of the electrified point.

Also by inversion find the distribution on an insulated charged conductor formed by the outer surfaces of two spheres cutting orthogonally, and if R and r be the radii of the spheres, whereof r is small in comparison with R, prove that the charge is distributed on the two surfaces in the ratio $4R^2 - 3r^2 : 3r^2$.

Tripos, 1880.

159. Prove that the repulsion between two equal spheres in contact, charged to potential V, is $\frac{1}{6} V^2 (\log 2 - \frac{1}{4})$.

Jesus College, 1879.

160. Two spherical conductors which have A and B for their common inverse points form an electric accumulator, the dielectric being such that its S. I. C. varies inversely as the square of the distance from A: prove that, when a charge is given to the accumulator, the equipotential surfaces are a system of spheres also having A and B for inverse points, and that the density of the electrification at any point P in either conductor varies as AP^{-4}.

Shew also that the capacity of the accumulator is $AB \, . \, K\left(\frac{f}{a} \sim \frac{g}{b}\right)^{-1}$, where a, b are the radii of the conductors, f, g the distances of their centres from A, and K is what the S. I. C. would be at B.

Tripos, 1882.

161. A number of equal spherical conductors are ranged symmetrically round a circle and an equal number of the same size are in like manner ranged symmetrically round the same circle: shew that if all the conductors receive equal charges, a solution by means of a finite system of images can be found for the potential at any point of space.

If the second series of conductors be connected by a suitable framework moveable about the centre of the circle, the first series being fixed, prove that the position of equilibrium is stable, and shew how the charge may be expressed in terms of the time of a small oscillation.

Trinity College, 1882.

162. Shew that the density of electricity at a point P of the surface of an insulated spherical conductor influenced by a quantity E placed on a concentric ring of radius a, is

$$\frac{E}{4\pi ar}-\frac{(a^2-r^2)\,E}{2\pi^2 r\,.\,PA^3}\left\{F(c)+\frac{a^2+r^2}{2r^2}\,cF'(c)\right\},$$

where r is the radius of the sphere, A the point of the ring most distant from P, $F(c)$ the complete first elliptic integral to modulus c, and $c^2=\frac{4a\rho}{PA^2}$, ρ being the distance of P from the axis of the ring.

Trinity College, 1880.

163. Shew that the density of electricity at a point P on the surface of an insulated spherical conductor influenced by a quantity Q on a concentric ring is

$$\frac{Q}{4\pi ar}-\frac{(a^2-r^2)\,Q}{2\pi^2 r\,.\,PA\,.\,PB^2}\,.\,E(c),$$

where a is the distance of the ring from the centre of the sphere whose radius is r, A is the point of the ring most distant from P, B the nearest point, $E(c)$ the complete second elliptic integral to modulus c, and

$$c^2=\frac{PA^2-PB^2}{PA^2}.$$

Trinity College, 1881.

164. If a uniform circular wire charged with electricity of line density $-e$ is presented to an uninsulated sphere of radius a, the centre

of which is in the line through the centre and perpendicular to the plane of the circular wire, prove that the electrical density induced at any point of the sphere whose angular distance from the axis of the ring is θ, is

$$\frac{f^2-a^2}{\pi a}\cdot\frac{Eef\sin\alpha}{\{a^2-2af\cos(\theta+\alpha)+f^2\}^{\frac{1}{2}}\{a^2-2af\cos(\theta-\alpha)+f^2\}},$$

where f is the distance of any point of the ring from the centre of the sphere, α is the angle subtended at the centre by any radius of the ring, and

$$E=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\phi}\,d\phi,\quad k^2=\frac{4af\sin\alpha\sin\theta}{a^2-2af\cos(\theta+\alpha)+f^2}.$$

Tripos, 1879.

165. Four hollow copper spheres A, B, C, D, each one decimetre in diameter, are suspended by silk strings: A and C have small holes at each extremity of their vertical diameters. B and D have similar holes at the upper extremities only and are suspended so that the centre of B is vertically below that of A and the centre of D vertically below C. Two copper wires permanently connect A with D and B with C and the distances between the spheres are such that their direct action on each other may be neglected. Two pipes which convey water from the same tank dip into A and C without touching them, so that spherical drops of water one millimetre in radius are formed at their extremities within the sphere and fall singly through the holes beneath into the spheres B and D respectively. If, before the water is turned on, A and C are charged to potentials V and $-V$ respectively by means of a battery and the battery wires are then removed, shew that when B and D are full of water the potential of A will be between $10^{591}V$ and $10^{535}V$, no electricity or water being supposed to escape, it being given that

$$\log_{10}101=2\cdot0043214 \text{ and } \log_{10}102=2\cdot0086002.$$

S. John's College, 1879.

166. A surface is formed by the revolution of a circle of diameter d about a tangent line: if a conductor of this shape be uninsulated and an electric point be embedded in the singular point but insulated

from the conductor, shew that the charge on the conductor outside any sphere whose centre is the singular point and radius $d\sin\phi$ varies as $\log\left(\tan\frac{\phi}{2}\right)$.

S. John's College, 1879.

167. An insulated hollow conductor is in the form of the surface generated by the revolution of the curve $r=a\cos^2\frac{\theta}{2}+b\sin^2\frac{\theta}{2}$ about the initial line, a and b being positive. At the pole is placed a charge E: shew that the surface density of the induced electricity at any point whose co-ordinates are (r, θ) is given by

$$4\pi\sigma=\frac{-(a-b)E}{\log\frac{a}{b}}\cdot\frac{1}{r^2\left(a^2\cos^2\frac{\theta}{2}+b^2\sin^2\frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

Tripos, 1880.

168. In the quadrant electrometer suppose that the quadrants and needle, or bodies equivalent to them, are of any form whatever, provided only that no part of the needle is very near to either quadrant or the case of the instrument. In a first experiment, the needle being charged, it is found that the needle is not deflected when the quadrants are charged to any equal potentials. Prove that in any other experiment in which the deflection is small, the deflection

$$=(A-B)(cC-aA-bB),$$

where A, B and C are the potentials of the quadrants and of the needle, and a, b, c constants. Hence prove that in Sir W. Thomson's instrument the deflection is

$$c(A-B)\{C-\tfrac{1}{2}(A+B)\}.$$

Tripos, 1879.

ELECTROKINETICS.

Thermo-Electricity, &c.

169. If in a thermo-electric circuit we neglect the heat dissipated in heating the wires, and find that keeping the upper junction at 100^0 C., and the lower in ice at 0^0 C., 4 oz. of ice are melted and 10000 footpounds of work done, how many degrees below the centigrade zero is the absolute zero? (Take 79 for the heat of melting of ice.)

S. Peter's College, 1878.

170. Wires of three different metals A, B, C, having resistances a, b, c, have their ends soldered together at two junctions which are maintained at (different) constant temperatures. If I_a be the strength of the current when A is cut, I_b the strength when B is cut, shew by a rigorous method that the strength of the current in C when all three wires are continuous is

$$\frac{a(b+c)I_a+b(a+c)I_b}{ab+ac+bc}.$$

Tripos, 1875.

171. Prove that if a current I pass at the junction of two metals, a quantity of heat $aIt(t_0-t)$ will be liberated or absorbed per unit of time, where t is the temperature of the junction.

Smith's Prize, 1879.

172. If p grammes of zinc be consumed in a galvanic battery per unit time when no external work is done by the current, and if q grammes be consumed when W units of external work are done per unit time, prove that

$$W=Eq\left(1-\frac{q}{p}\right),$$

where E is the energy developed by the battery per gramme of zinc consumed.

Tripos, 1880.

173. A thin wire is interpolated in a galvanic circuit of known resistance: find the resistance of the interpolated wire in order that the quantity of heat generated in it in a given time may be a maximum.

Smith's Prize, 1874.

174. The resistance of a battery with its electrodes is R, and its electromotive force is constant. The circuit is completed by a fine wire of uniform section, whose resistance is a function of its temperature, which is supposed to be the same at all points of the wire. The wire is such that if no heat were generated in it, it would lose one per cent. of its excess of temperature over that of the air in a time T: and the E.M.F. is such that if the wire were prevented from losing heat, its resistance would increase one per cent. in a time τ. Shew that if r is the resistance of the wire when the current is in equilibrium, the equilibrium will be unstable if

$$(R-r)\,T > (R+r)\,\tau.$$

Smith's Prize, 1879.

Linear Conductors.

175. A frame of straight wires is in the form of the sides of a regular hexagon and its three diagonals: if a current of electricity enters at any angular point and leaves at the diagonally opposite point, find what length of wire would have as much heat generated in it per unit of time as the frame.

Trinity College, 1877.

176. Let AB, BC, CD, DA be four uniform wires, each of unit resistance, joined in the form of a square $ABCD$: let E be a point in the side CD such that CE is to ED in the ratio of $\sqrt{2}$ to 1: and let A be joined to E by a wire AE of unit resistance. Shew that, if the points A and C be maintained at different potentials, then the potential of B is equal to that of E, so that no current will flow along a wire joining BE.

Tripos, 1882.

177. A tetrahedral framework is made of wires cut from the same coil: if pairs of opposite edges be equal and of lengths a, b, c respec-

tively and a current enters and leaves the framework at the ends of an edge of length a, then the strengths of the currents in the pairs of edges of length a are in the ratio

$$b(a+c)+c(a+b)\ :\ b(a+c)-c(a+b).$$

Jesus College, 1880.

178. In the previous question shew that the resistance of the whole framework is that of a length of wire equal to

$$\tfrac{1}{2}\left(\frac{ab}{a+c}+\frac{ac}{a+b}\right).$$

S. John's College, 1880.

179. Four points A, B, C, D are joined in pairs by wires, whose resistances are proportional to the distances of the points joined, and electromotive forces E_1, E_2, E_3 act in the wires joining BC, CA, AB, prove that if the resulting current in DC be independent of E_3, it will be $\dfrac{E_1a'-E_2b'}{aa'+b'c'+c'a'+a'b'}$ where $BC=a$, $DA=a'$, &c.: and if A, B, C be fixed the locus of D is a sphere of radius $\dfrac{abc}{a^2-b^2}$.

Tripos, 1883.

180. One side of the quadrilateral of the Wheatstone's Bridge is formed of a condenser of very great capacity, which is gradually charged from the battery. Given the several resistances, that of the wires leading to the coatings of the condenser being insensible, compare the potential of the condenser at the moment when there is no current in the bridge with that of the battery supposed unconnected.

Tripos, 1878.

181. It is proposed to construct a resistance coil the percentage error of which shall be a minimum: the probable error arising from imperfect connexion of the electrodes is r, and the defect of insulation is such that independently of the wire the conductivity between the electrodes is C, with a probable error c: shew that the best value for the resistance of the wire is such that if x is the actual resistance of the apparatus

$$x^4c^2=(1-Cx)^3(1+Cx)r^2.$$

Smith's Prize, 1879.

Cables and Condensers.

182. A submarine cable was insulated at the distant end, charged, and its fall of potential noted at the near end, on a Thomson's Quadrant Electrometer whose zero was 360. The following readings of the scale of the electrometer were made at intervals of half a minute: 659, 656, 654, 652, 650, 648, 646, 643, 641, 639, 637. Calculate the rate of loss of potential and the insulation resistance per knot, the capacity per knot being ·404 Farad.

S. Peter's College, 1878.

183. A submarine telegraph cable 1885 miles long is formed of a copper conductor $\left(\frac{1}{2e}\right)$ inches in diameter surrounded by a gutta-percha coating $\frac{1}{2}$ in. in diameter. The S.I.C. of gutta-percha being 4·2, shew that the capacity of the cable is equal to that of a sphere of the same size as the earth.

S. John's College, 1878.

184. Prove that R the resistance offered by the sheath of a cable to a lateral discharge through it is given by the formula

$$R = \frac{t}{C \log \frac{P}{P'}},$$

where C is the capacity of the cable, and t the time in which its potential falls from P to P' by discharge across the sheath.

Jesus College, 1878.

185. The wire of a galvanometer has the same resistance as that used for a telegraph line. At 100 miles from the battery the needle is deflected through a certain angle, the wire in the coil making 10 turns. The galvanometer being removed to 250 miles from the battery and the number of turns in the coil doubled, the deflection is the same as before. Prove that the internal resistance of the battery is equal to 50 miles of wire: and that to produce the same deflection at 610 miles the wire in the coil must make 44 revolutions.

Trinity Hall, 1879.

186. If A and B be the ends of a wire with one fault, and R, S the resistances to a current sent from A when B is insulated and to earth respectively; and if R', S' be the corresponding resistances to a current sent from B, prove that $R : S :: R' : S'$.

Prove also that the same result will be true if there are two faults.

Tripos, 1878.

187. The conduction resistance of a cable 1000 miles long is 10 ohms per mile, whilst the insulation resistance is 10 megohms: if the sending end be at given potential and the receiving end to earth, find the whole charge of the cable when a steady current passes through it. Shew also that if the cable have a leakage fault at the middle point the resistance of which is equal to that of a length a miles of the cable the strength of a steady current at the receiving end will be lowered in the ratio

$$1 : 1 + \frac{500}{a} \cdot \frac{\epsilon - 1}{\epsilon + 1},$$

where ϵ is the base of the Napierian logarithms.

Tripos, 1880.

188. Investigate the electrostatic capacity per unit of length of a submarine cable, the diameter of the core being d, and the external diameter of the insulating sheath D.

Assuming the leakage to bear at all points the same ratio to the charge, form the equation for the transmission of electric potential along the cable: and shew from it that, *ceteris paribus*, the time necessary for a definite electrical operation is as the square of the cable's length. If the leakage be considerable, how must the battery-power depend on the length of the cable in order that slow signals may be of a given intensity?

Tripos, 1875.

189. If C be the capacity of a condenser, and R the resistance of the medium interposed between its surfaces, shew that the time of falling from charge to half charge is $CR \log_e 2$. Shew that in the case of a cable whose capacity per mile is given this is a convenient way of testing the insulation resistance per mile.

S. Peter's College, 1880.

190. A Leyden jar loses $0{\cdot}000001$ of its charge per second by conduction through the glass. The specific inductive capacity of the glass of which it is made is 8. Shew that the resistance of a cubic centimetre of the glass is roughly 14×10^{17} ohms, having given that one electro-magnetic unit of quantity is equal to 3×10^{10} electrostatic units and one ohm is equal to 10^9 electro-magnetic units.

Tripos, 1879.

191. The outer coatings of two condensers A and B are put to earth and their inner coatings are connected through a galvanometer whose resistance is 4000 ohms. The capacity of A is three microfarads, that of B is one microfarad, and the two condensers are charged to potential one volt. The inner coatings of A and B are then put to earth simultaneously through resistances of 1000 and 2000 ohms respectively. Shew that the whole amount of electricity which will flow through the galvanometer is equal to $\frac{1}{7}$ of the charge of the smaller condenser.

S. John's College, 1879.

192. A telephone is attached to one extremity of a long and perfectly insulated cable, and is regarded as an instrument for producing a harmonically varying potential at one end of the cable or detecting such variation at the other end; its resistance and self-induction being neglected. If different notes be sounded, prove that the amplitude of the electrical oscillation will be diminished in any given proportion at distances inversely as the square root of the pitch of each note. Prove also that different cables will be equivalent for the purposes of the telephone if the product of the capacity and the resistance per unit of length be the same in each.

Tripos, 1879.

Conduction in two and three dimensions.

193. On an infinite conducting plane there are at a distance $2c$ from each other two sinks; and a source of electricity half way between them, the intensities of all three being the same, prove that the lines of flow are given by the equation

$$x(x^2+y^2-c^2)=my(x^2+y^2+c^2),$$

and find the equipotential curves.

S. John's College, 1878.

194. In an infinite plate bounded by a straight edge, if one electrode be on the edge and the other at a point on the plate, such that the line joining the electrodes is perpendicular to the edge, prove that the current lines are lemniscates.

S. John's College, 1877.

195. A circular disc has a source at its centre and two equal sinks at the ends of a diameter. What are the lines of flow?

S. John's College, 1878.

196. A current of electricity enters at a point A on the rim of a conductor which is a circular disc and leaves at the centre. If the disc be of uniform specific conductivity prove that the equation of any stream line referred to A as origin is

$$r=a\,.\,\frac{\sin(\alpha+2\theta)}{\sin(\alpha+\theta)},$$

where a is the radius of the disc and α is constant.

Trinity College, 1877.

197. The resistance of a circular plate touched by two electrodes at A and B is

$$\lambda\log\frac{BA^2\,.\,AB'\,.\,A'B}{\rho^2\,.\,AA'\,.\,BB'},$$

where A' and B' are the "images" of A and B and ρ is the radius of the small wire.

Trinity College, 1879.

198. Shew that a system of circles with the same radical axis is a possible form of equipotential curves in a plane electric field in which the flow is steady, and find the lines of flow.

Prove that the E.M.F. at any point varies inversely as the product of the distances of the point from the limiting points of the system.

S. John's College, 1878.

199. Verify that if the electrodes of a current C be placed at opposite corners of a uniform conducting rectangle

$$\phi = \frac{RC}{\pi} \log \frac{1 - \mathrm{cn}^2\left(K\frac{x}{a},\, k\right)\mathrm{cn}^2\left(K'\frac{y}{b},\, k'\right)}{k^2\,\mathrm{cn}^2\left(K\frac{x}{a},\, k\right) + k'^2\,\mathrm{cn}^2\left(K'\frac{y}{b},\, k'\right)},$$

$$\psi = \frac{2C}{\pi} \tan^{-1} \frac{\mathrm{sn}\, K'\left(\frac{y}{b},\, k'\right)\mathrm{dn}\left(K'\frac{y}{b},\, k'\right)\mathrm{cn}\left(K\frac{x}{a},\, k\right)}{\mathrm{sn}\left(K\frac{x}{a},\, k\right)\mathrm{dn}\left(K\frac{x}{a},\, k\right)\mathrm{cn}\left(K'\frac{y}{b},\, k'\right)},$$

where $\frac{K}{K'} = \frac{a}{b}$: a and b being the sides of the rectangle, R the specific resistance of the plate, and the axes of co-ordinates being two adjacent sides of the rectangle meeting in an electrode.

Tripos, 1878.

200. A system of electrodes of equal strength 2π is placed at the angular points of a system of rectangles the lengths of whose sides are $2a$, $2b$, the electrodes at the extremities of one diagonal being positive those at the extremities of the other both negative. The centre of one rectangle being taken as origin and the co-ordinate axes being parallel to the sides of the rectangles, prove that the potential and current functions will be respectively

$$\phi = \log \frac{\mathrm{dn}\,(x, k)\,\mathrm{dn}\,(y, k') - kk'\,\mathrm{sn}\,(x, k)\,\mathrm{sn}\,(y, k')}{\mathrm{dn}\,(x, k)\,\mathrm{dn}\,(y, k') + kk'\,\mathrm{sn}\,(x, k)\,\mathrm{sn}\,(y, k')},$$

$$\psi = 2 \tan^{-1} \frac{k'\,\mathrm{cn}\,(y, k')}{k\,\mathrm{cn}\,(x, k)},$$

the moduli k, k' being determined by the equations

$$b\int_{\alpha}^{\frac{\pi}{2}} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}} = a\int_{0}^{\frac{\pi}{2}} \frac{d\theta}{(1-k'^2\sin^2\theta)^{\frac{1}{2}}},$$

$$k^2+k'^2=1.$$

Prove also that the potential function ϕ may be expressed by the series

$$\Sigma_{i=0}^{i=\infty} \frac{(-1)^i}{2i+1} \cdot \frac{\sinh (2i+1)\frac{\pi y}{2a}}{\cosh (2i+1)\frac{\pi b}{2a}} \sin (2i+1)\frac{\pi x}{2a}.$$

Smith's Prize, 1882.

201. Obtain a solution of the equation $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0$ with two arbitrary functions of x, expanded in positive integer powers of y. Apply the solution to find the potential of electricity flowing along a lamina of metal bounded by the curves $\pm y=F(\omega x)$ where ω is small, so that y varies slowly with x. Shew that the resistance of such a lamina bounded by two equipotential lines cutting the axis of x in $x=a$ and $x=b$ is very nearly the integral between these limits of

$$\frac{1}{F}+\frac{F^2}{6}\cdot\left[\frac{d^2}{dx^2}\cdot\frac{1}{F}+\tfrac{1}{6}\cdot\frac{d^2}{dx^2}\left(F^2\frac{d^2}{dx^2}\frac{1}{F}\right)\right]-\frac{F^4}{120}\frac{d^4}{dx^4}\cdot\frac{1}{F},$$

if the lamina be of unit specific conductivity.

202. The points on a thin spherical current sheet of equal resistance σ in all parts being defined by the longitude ϕ, and the latitude χ: there is at the point $\phi=\alpha$, $\chi=0$ a source of electricity of strength E: and at $\phi=-\alpha$, $\chi=0$ a sink of equal strength. Shew that the potential at any point of the sheet is

$$V=\frac{\sigma E}{4\pi}\log\frac{\cosh\mu-\cos(\phi+\alpha)}{\cosh\mu-\cos(\phi-\alpha)}+a\text{ const.},$$

where
$$\mu=\log\frac{1+\sin\chi}{\cos\chi}.$$

Trace completely the equipotential curves and express their equation in terms of the angular distances from the source and sink.

S. Peter's College, 1879.

203. Verify that the electric potential at any point P of a solid homogeneous sphere of resistance k, due to a current of intensity I flowing through electrodes at A, and B the ends of a diameter, is

$$\frac{Ik}{2\pi}\left(\frac{1}{AP}-\frac{1}{BP}-\frac{1}{AB}\log\frac{AN+AP}{BN+BP}\right),$$

where N is the foot of the perpendicular from P on the diameter AB.

Tripos, 1881.

204. Two families of orthogonal surfaces are formed by the revolution round the axis of x of the two families of circles

$$x^2+y^2+2ay\tan\theta=a^2, \text{ and } x^2+y^2-2ax\sec\phi=-a^2.$$

The annulus between ϕ and $\phi+\delta\phi$, and the shorter arcs of θ and $\theta+\delta\theta$ is filled with conducting material whose specific resistance is ρ. Shew that the resistance to a current flowing from ϕ to $\phi+\delta\phi$ is

$$\frac{\rho}{2\pi a}\cdot\frac{1+\sin\theta\sin\phi}{\sin^2\phi\cos\theta}\cdot\frac{\delta\phi}{\delta\theta}.$$

Hence shew that if R be the resistance of the sphere $\theta=0$ with equal electrodes of no resistance of the form given by ϕ_0 and $\pi-\phi_0$, then

$$R>\frac{\rho}{\pi a}\int_{\phi_0}^{\frac{\pi}{2}}\frac{d\phi}{\sin\phi\log(1+\sin\phi)}<\frac{\rho}{\pi a}\,\frac{\log\cot\frac{1}{2}\phi_0}{\log\left\{1+\tan\phi_0\log\cot\frac{\phi_0}{2}\right\}}.$$

Hence shew further that

$$R>\frac{\rho}{\pi a}\cot\phi_0<\frac{\rho}{\pi a}\cot\phi_0\left(1+\tfrac{1}{2}\tan\phi_0\log\cot\frac{\phi_0}{2}\right).$$

Tripos, 1883.

205. A galvanic current is lead through a liquid in a rectangular metallic cell, which itself forms one electrode, the other being a wire only just introduced beneath the surface: determine the potential at any point of the liquid.

How may the forms of the equipotential surfaces be explored experimentally?

Smith's Prize, 1880.

MAGNETISM.

206. Shew that a magnet held in any position in a field of uniform force experiences a couple having the same moment and axis as that which would be experienced if the "imaginary magnetic matter" representing it were collected in two points, the northern all in the C.G. of the northern imaginary matter, and the southern all in the C.G. of the southern.

Tripos, 1874.

207. Two magnetic particles of moments m and m' are fixed at two corners of an equilateral triangle with their axes bisecting the angles. A third magnetic particle is free to move at the other angular point. Shew that its axis makes with the bisector of the third angle an angle

$$\tan^{-1}\frac{\sqrt{3}}{7}\,\frac{m \sim m'}{m+m'}.$$

Tripos, 1882.

208. Two magnetic molecules are free to move about parallel axes through their middle points: shew that there are eight positions of equilibrium, four in which the magnets lie in the plane through the axes, and four in which the magnets are perpendicular to this plane. Prove that if the angle between the axes and the line joining the centres of the magnets be greater than $\sin^{-1}\sqrt{\frac{2}{3}}$, of the first four positions of equilibrium two are stable and two unstable, and in the last four the equilibrium is stable-unstable.

Tripos, 1877.

209. One small magnet is fixed and the centre of another under its influence is moved in such a manner that its axis, free to turn in any direction, always points to a fixed point in the axis of the first produced. Prove that the centre of the moveable magnet always lies on a fixed sphere.

Clare College, 1878.

210. Prove that, if A, B be the poles of a uniformly magnetized infinitely thin magnet, the strength of whose poles is m, the vector potential at any point P is

$$\frac{m}{p}[\cos PAB + \cos PBA],$$

where p is the perpendicular on the line joining the poles of the magnet. Shew also that the direction of the vector potential is perpendicular to the plane PAB.

Tripos, 1882.

211. Shew that for a simple magnetic shell in the form of a circle, the direction of the vector potential at any point is perpendicular to a plane through the point and a normal to the plane of the shell through the centre.

S. Peter's College, 1881.

212. In the case of a small mass of an isotropic substance, whose susceptibility to magnetic induction is feeble, placed in a field of force due to magnets at a considerable distance, prove that the directive tendency is very small as compared with the tendency to motion of the centre of mass.

Tripos, 1880.

213. Prove that in a uniform field the magnetic force in the interior of a straight tube of soft iron perpendicular to the axis of the tube is

$$\frac{4(1+4\pi k)b^2}{(1+4\pi k)^2(b^2-a^2)+2(1+4\pi k)(b^2+a^2)+b^2-a^2}$$

of the external magnetic force in the same direction, a and b being the radii of the cylindrical surfaces, and k the coefficient of induced magnetization.

Jesus College, 1881.

214. At a point in a magnetized body where the intensity of magnetization is I, a small sphere is scooped out: find the force at the centre due to the surface distribution.

S. Peter's College, 1879.

215. A small magnet is placed at the centre O of a spherical cavity made in a large mass of soft iron: prove that the direction of

the induced magnetism at any point P makes an angle $\tan^{-1}(\frac{1}{2}\tan\theta)$ with OP, OP making an angle θ with the axis of the magnet.

Clare College, 1881.

216. Shew how the determination of the magnetism induced in a homogeneous isotropic body acted on by external magnetic forces may be reduced to a mathematical problem.

Clare College, 1879.

217. Shew that the potential at any point in soft iron with a plane face in front of which is placed a magnetic molecule is $\frac{1}{1+2\pi\kappa}$ of the potential if the iron were removed, where κ is the coefficient of magnetic induction.

Clare College, 1879.

218. A permanent magnet in the form of a rectangular parallelepiped, $ABCD$, and $A'B'C'D'$ being opposite faces at a distance c apart, has uniform magnetization I parallel to AB or $A'B'$. Shew that if a uniform torsion be given, $A'B'C'D'$ being turned through an angle α about its centre, the potential may be obtained by supposing a distribution of matter over the faces $BCC'B'$ and $ADD'A'$ whose surface density is $I\left(1+\alpha^2\frac{r^2}{c^2}\right)^{-\frac{1}{2}}$, r being the distance from the median line of the face, measured parallel to the base.

Tripos, 1883.

219. Prove that when, as in iron, the magnetic susceptibility is 30 or upwards, the intensity of magnetization in a sphere, and still more in an oblate ellipsoid of revolution (when the lines of force of the magnetic field are parallel to its axis) is nearly the same as if the susceptibility were infinite: but that in a very long prolate spheroid it is nearly proportional to the susceptibility.

Tripos, 1875.

220. A magnetic molecule has its centre fixed at a distance c from the centre of a uniformly magnetized sphere of radius a, and can oscillate in the plane through the axis of the sphere and its own axis; shew that the time of a small oscillation is

$$\left\{\frac{a^3mI}{3\pi Mk^2c^3}\sqrt{1+3\cos^2\theta}\right\}^{-\frac{1}{2}},$$

where θ is the angle the line joining the centres makes with the axis of magnetization of the sphere, I the intensity of magnetization of the spherical magnet, Mk^2 the moment of inertia of the molecule about its centre, and m its magnetic moment.

Tripos, 1881.

221. Find the magnetization of a solid ellipsoid of soft iron placed in a field of uniform and parallel magnetic force.

Shew how to solve the same problem when the ellipsoid has a confocal cavity, considering for simplicity the magnetic force to be parallel to an axis.

Tripos, 1881.

222. In the case of symmetry round a line, if the magnetic inductive capacity vary as the pth power of the distance from a point in that line, the magnetic potential will be of the form $r^n S_i$, where S_i is a surface harmonic of order i and

$$i(i+1)=n(n+p+1).$$

Tripos, 1882.

223. Work out the analogy between a sphere in a liquid and a uniformly magnetized sphere, pointing out the vortices in the first case and the electric currents in the second, which produce the same effect.

Tripos, 1881.

224. How might the dipping needle be employed by itself for finding the direction and intensity of the terrestrial magnetic force at any locality the magnetic meridian of which is approximately known?

Smith's Prize, 1873.

225. Calculate the disturbances produced on each other by a variation-compass and a dip needle whose relative position is given.

Tripos, 1875.

226. Mention the different kinds of variation of the magnetic elements which are found to be indicated by discussions of magnetic observations made continuously at a given locality.

Smith's Prize, 1873.

227. It has been observed that the resultant horizontal component of the magnetic force on the compass of a certain ship is constant for all courses: prove that this compass is free from semicircular and quadrantal deviation, but that it may have a constant deviation.

Tripos, 1879.

ELECTROMAGNETISM.

Statics.

228. Assuming that the action of a magnetic pole on an element of an electric circuit varies as the sine of the angle between the distance of the pole from the element and the direction of the element directly, and as the square of that distance inversely, and is perpendicular to the plane through the pole and the element, prove that the action of such a pole on any closed circuit is in all respects the same as it would be upon an infinitely thin magnetic shell bounded by the circuit.

Tripos, 1877.

229. A horizontal current of indefinite length and known intensity is situated in the plane of the magnetic meridian, and a needle of given length, free to move in a horizontal plane about an axis through its centre, is placed with that centre in the same magnetic meridian with the current, and at a given distance from it. Find an equation for determining the deviation produced in the needle by the current.

Prove that if the distance of the centre of the needle from the current be such that the deviation of the needle is a maximum for a given intensity of the current, the plane through the current and either pole of the needle will make an angle of 45^0 with the horizon.

Tripos, 1877.

230. A rectangle of wire carrying a current is hung up in a given vertical plane, and one of its vertical sides is capable of sliding along the two horizontal sides: find the force necessary to keep that side from sliding.

Trinity College, 1877.

231. Prove that a current of strength i in the axis of a hyperboloid of revolution exerts on an element ds of a current of strength i' in any one of the generating lines of the hyperboloid an electromagnetic force in the direction of the normal to the hyperboloid at the element of intensity

$$\frac{2ii'\,ds}{r^2}\sqrt{r^4 - a^2\sin^2\alpha}$$

where r is the distance of the element from the axis of the hyperboloid and α the angle, and a the shortest distance between the generating line and the axis.

Tripos, 1881.

232. Prove that the potential of a circular current is the same as that of a certain imaginary rod lying along its axis; and investigate a similar expression for the potential of a homogeneous spherical shell.

Smith's Prize, 1881.

233. Given a simple magnet moveable about its C.G. and an infinite straight current: find their relative positions for stable, unstable, and neutral equilibrium, and the time of a small oscillation about the position of stable equilibrium.

S. Peter's College, 1879.

234. A current of strength i circulates in a fixed circular wire of radius a. At a point on the axis of this circle a magnetic molecule of moment m has its centre fixed and is free to turn about it: shew that when its axis makes an angle θ with that of the current, the moment of the forces tending to diminish θ is

$$\frac{2\pi}{a} mi \sin^3\alpha \sin\theta,$$

where α is the angle subtended at the centre of the molecule by any radius of the circle.

Tripos, 1882.

235. In the case of two circular circuits in planes at right angles to the line joining their centres, prove that

$$M = 2\pi(a^2 - c^2)\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta\, d\theta}{\sqrt{a^2\sin^2\theta + c^2\cos^2\theta}},$$

where a and c are the longest and shortest straight lines respectively which can be drawn from one circuit to the other.

Find the mutual force between the circuits.

Tripos, 1874.

236. A sphere of soft iron of radius a is placed with its centre at the centre of a circular current of radius c. If powers of $\frac{a}{c}$ above the third be neglected, the magnetization at any point of the sphere makes with the axis of the current an angle

$$\tan^{-1}\left(\tfrac{7}{2}\,\frac{3+4\pi\kappa}{7+12\pi\kappa}\cdot\frac{r^2}{c^2}\sin\theta\cos\theta\right),$$

where κ is the coefficient of induced magnetization.

Tripos, 1881.

237. A plane circular wire can move about a diameter which is parallel to an infinite straight wire whose distance from the centre of the circular wire is equal to the radius of the latter. Shew that when currents of intensities I_1, I_2 are passed through the wires the couple tending to turn the circular wire about its fixed diameter is

$$4\pi I_1 I_2 c\left(\cos\phi - \frac{\cos\frac{3}{2}\phi}{\sqrt{2\cos\phi}}\right),$$

where c is the radius of the circular wire, and ϕ the angle between a normal to its plane and the above shortest distance, such that

$$\frac{\pi}{2} > \phi > -\frac{\pi}{2}.$$

Tripos, 1881.

238. Two circular currents are in one plane: their radii are a and A, and the distance between their centres is b, which is large in comparison with the radii: prove that the resultant force between the currents is approximately

$$\frac{3\pi^2 a^2 A^2}{b^4}\,ii'.$$

Tripos, 1880.

239. Two circular conducting circuits A and B, similar in all respects, are fixed in the position of the ends of a right cylinder, and a

constant E. M. F. begins to act on A: prove that at the time t the induced current in B and the current in A in the opposite direction are in the ratio $e^{-\alpha t}-e^{-\beta t} : 2-e^{-\alpha t}-e^{-\beta t}$, where $\alpha=\frac{R}{L-M}$, $\beta=\frac{R}{L+M}$, R is the resistance in each, and LM are the coefficients of self- and mutual-induction.

Tripos, 1883.

240. A straight wire AB is to have its ends connected by a bent wire of given length so as to form a conducting circuit. Shew that in order that the sudden generation of a current of electricity in an infinite wire parallel to AB may induce in the circuit the greatest possible current, it must have the form determined by the intrinsic equation

$$\rho=Ae^{k\cos\psi}.$$

Determine the form of the wire when the area contained by the circuit is given.

Tripos, 1883.

241. The space between a plane electric circuit and one of its equipotential surfaces which meets it at an angle ω is filled with soft iron of specific inductive capacity c: prove that the solid will be magnetized by a unit current as a lamellar magnet whose total strength

$$=\frac{2c\omega}{1+4c\pi-2c\omega}.$$

Tripos, 1879.

242. An infinitely long linear conductor spreads out at the point A into a thin sheet, having the form of a circular sector of angle $2a$ and infinite radius. If the prolongation of the linear conductor bisects the sector, shew that the force on a magnetic pole of unit strength placed on a line through A perpendicular to the plane of the sector and distant d centimetres from A is

$$\left\{1+\frac{\sin a}{a}\right\}\frac{C}{d},$$

where C is the strength of the current in the linear conductor measured in electromagnetic (C. G. S.) units.

Corpus College, 1877.

243. A circular wire centre O is in the magnetic meridian and a small magnet is moveable about a vertical axis at C, OC being perpendicular to the plane of the wire. If a transient current be passed through the wire, and θ be the elongation of the first vibration of the magnet, prove that

$$\sin\frac{\theta}{2}=\sin^3 a\sin\frac{\theta'}{2},$$

where θ' is the value of θ when the magnet is at O and a is the angle subtended by the radius of the wire at C.

244. A magnetic pole is moving with constant velocity from an infinite distance along the axis of a circular conducting ring. If W be the work done on the magnetic pole and H the heat generated up to any time t, shew that

$$W=J\left(H+\frac{I}{2R}\cdot\frac{dH}{dt}\right),$$

where I is the coefficient of self-induction and R the resistance of the ring.

S. John's College, 1880.

Kinetics.

245. If M denote the amount of induction in a magnetic field through a circle of radius ρ, whose axis is the co-ordinate axis z and whose centre is at a distance z from the origin, prove that M satisfies the differential equation

$$\frac{d^2M}{dz^2}+\frac{d^2M}{d\rho^2}-\frac{1}{\rho}\frac{dM}{d\rho}=0.$$

If the field is symmetrical in regard to the axis of z, prove that at any point not in the substance of magnets or of wires carrying electric currents the vector potential is $\dfrac{M}{2\pi\rho}$.

From the general equations of electromotive force deduce the differential equation for the variation of the electric current in a fixed fine circular wire due to the motion of a magnetic molecule in its axis, the axis of the molecule being in the axis of the circuit.

Tripos, 1880.

246. A circuit is placed on a parallel of latitude; if it contracts so as to have always the form of a small circle touching the parallel of latitude on which it originally lay, and so that its pole moves from the pole of the earth with a constant velocity, prove that when the radius is ρ, and the latitude of the pole λ, the E. M. F. is proportional to

$$2(\sin\lambda\cos\rho+\cos\lambda\sin\rho)\sin\rho,$$

and is a maximum when

$$2\sin\lambda=9\sin(\lambda+2\rho)+\sin(\lambda-2\rho).$$

Trinity College, 1880.

247. If a circular coil of given radius, length, and resistance revolves uniformly in a given uniform magnetic field, find the current passing through it.

Trinity College, 1876.

248. A copper wire in the form of a complete circle is suspended so that it can turn without friction about a diameter which is perpendicular to the direction of the earth's magnetic force: it is set in rotation and then left free, determine its subsequent motion, neglecting the self-induction of the wire.

Tripos, 1878.

249. A copper disc in a uniform field of magnetic force whose lines of force are perpendicular to its plane, is spun in its own plane round a fixed point O, and is continuously touched at two points by the fixed electrodes of a galvanometer. Shew that the current in the galvanometer is proportional to the difference of the areas swept out by the radii vectores from O to the points touched.

Tripos, 1883.

250. If a circular disc spin about a vertical axis, shew that the periphery of the disc will be at uniform electric potential, which differs from that at the axis by $\frac{1}{2}\omega a^2 K\sin\alpha$: where a is the radius of the disc, ω its angular velocity, K the intensity of the earth's magnetic force and α the dip.

If the disc spin in the opposite direction to the earth's rotation, is the potential at the periphery greater or less than at the axis?

Tripos, 1879.

251. If a circular plate oscillate normally in front of a magnetic pole A very near its centre C, prove that if the mutual induction of the currents be neglected these currents will be given by the equation

$$\frac{1}{PA} - \frac{fx}{a^3} = \text{constant},$$

where a is the radius of the disc, f the projection of CA upon it, x the abscissa of any point P of the disc measured along f.

If the pole of the magnet be in the centre of an induction coil, shew how to find the currents which will be induced in this coil, and shew that they are approximately of the same period as the vibrations of the plate.

Tripos, 1878.

252. A sphere of radius a charged with a quantity e of electricity moves in an infinite field with a velocity whose components parallel to the axes are p, q, r and the components of electro-kinetic momentum at a point distant R from the centre are

$$F = \frac{\mu e p}{R} + \frac{d\chi}{dx}, \quad G = \frac{\mu e q}{R} + \frac{d\chi}{dy}, \quad H = \frac{\mu e r}{R} + \frac{d\chi}{dz},$$

where

$$\chi = -\frac{\mu e}{2}\left(\dot{R} + a^2 \frac{d}{d\theta}\frac{1}{R}\right)$$

and

$$\frac{d}{d\theta} = p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}.$$

Shew hence that the components of the electric displacement at each point of the field are given by

$$\frac{df}{dt} = \frac{e}{4\pi} \cdot \frac{d}{dx} \cdot \frac{d}{d\theta} \cdot \left(\frac{1}{R}\right)$$

and two similar equations. Find also the electro-kinetic energy.

Trinity College, 1883.

Induction Coils: Machines.

253. Prove that in order to get the greatest possible induction from an induction coil of given external radius, the radius of the iron core, the outer radius of the inner coil, and the outer radius of the outer coil must be approximately as 2 : 3 : 4.

[Neglect effect of ends of coil and take sp. ind. capac. of iron large.]

Tripos, 1874.

254. An electric current of strength I traverses a wire coiled on a spherical core in such a way that the distances between successive planes of the coils is everywhere very small and constant; shew that the effect in external space is the same as of a small magnet whose magnetic moment varies jointly as the strength of the current and the volume of the sphere.

S. John's College, 1880.

255. A wire is coiled round a sphere so as to produce within the sphere a magnetic potential of the form of a solid zonal harmonic of the third degree: the number of windings is N, and the strength of the current is γ. S is the area of a plane curve bounded by a conductor, which is placed within the sphere in any plane perpendicular to the axis of the harmonics, and has a current strength γ' running through it. Prove that the coefficient of induction for the internal conductor is

$$M = \frac{24\pi}{7a^3} \cdot NS\left(z^2 - \tfrac{1}{2}k^2\right),$$

where k is the radius of gyration of S about the axis, and a is the radius of the sphere. Find the force on the conductor parallel to the axis, and shew that it is constant when z is constant.

Trinity College, 1879.

256. Prove that if an ellipsoid be coiled with wire in parallel equidistant planes, the magnetic force within it will be uniform if a uniform current flow through all the wires.

Trinity College, 1881.

257. Investigate a method of coiling fine wire on the surface of a sphere so that on passing a current through the coil it may produce a uniform field of magnetic force at all points within the surface of the sphere.

S. John's College, 1879.

258. The outer coatings AA' of two condensers, capacities CC', are connected with earth. The inner coatings BB' are connected through two resistances R, R' to a point P. B, B' are also connected through a galvanometer whose resistance is G. P is suddenly raised to potential E, and the galvanometer needle is deflected through an angle a. Shew that

$$\sin\frac{a}{2}=\frac{\pi k}{HT}\cdot\frac{E(R'C'-RC)}{G+R+R'},$$

H being the strength of the field in which the needle hangs, T the time of swing, and k the galvanometer constant.

Trinity College, 1881.

259. An electro-magnet is constructed by winding a wire uniformly round n long coaxial circular cylinders of soft iron of equal thickness and length, enclosing each other: the number of layers of wire between two adjacent cylinders being always the same and equal to the number of layers outside the last cylinder: shew how to find the magnetic moment of the combination. Solve fully the case when $n=2$.

Tripos, 1878.

260. A magneto-electric machine requires W units of work per second in order to drive it. When the total resistance of the coils of the machine and of the external circuit is R, prove that the electromotive force of the machine cannot exceed $\sqrt{WR}$.

Tripos, 1879.

261. Two magneto-electric machines with permanent inducing magnets are used for the transmission of power, the first being driven by an engine with uniform velocity, and producing a current which is passed through the second machine. The second machine acts as a magneto-electric engine. When the second machine is standing, but the current of the first is passed through its coil, the first machine is found to absorb a power W from the engine. Prove that, neglecting

the loss due to friction, to disruptive discharges, and to short circuiting of the coils, the maximum power obtainable from the second machine is $\frac{1}{4}W$, and that then the first machine absorbs $\frac{1}{2}W$ from the engine.

Tripos, 1879.

262. The horse power required to drive a magneto-electric machine in which the magnet is of constant strength is proportional to the speed when the terminals are insulated. The E. M. F. is proportional to the speed when the terminals are connected, whatever the external resistance may be. When driven at 40 revolutions per second, its E. M. F. is equal to that of 3 Grove's cells. The resistance of the machine itself is $\frac{1}{2}$ Ohm and the power required to drive it at 40 revolutions per second is $\frac{1}{80}$ H. P. when the terminals are insulated, and $\frac{1}{20}$ H. P. when they are connected by a wire of $\frac{1}{2}$ Ohm resistance. The resistance of a Grove's cell being $\frac{1}{6}$ Ohm, shew that if 3 cells are connected in series with the machine they will drive it at 20 revolutions per second, and that 6 Grove's cells arranged in series will drive it at 50 revolutions per second.

S. John's College, 1879.

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